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UNIVERSITY OF ALBERTA

AXISYMMETRIC PROBLEMS IN TENSION-FIELD THEORY

BY

XINNIAN LI

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

DEPARTMENT OF MECHANICAL ENGINEERING

Edmonton, Alberta

Fall 1994

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FACULTY OF GRADUATE STUDIES AND RESEARCH

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This dissertation is dedicated to my wife and my parents

Abstract

This work focuses on the wrinkling analysis of elastic membranes. It is demonstrated that certain finite deformations of the membrane often associate with wrinkling in a certain area of the deformed membrane. Deformation gradient suffers finite jumps of discontinuity across the boundaries separating the tense and wrinkled regions. Wrinkling analysis is taken into account by using the relaxed strain energy in the equations of equilibrium which, as a consequence, yields a uniaxial state of stress in wrinkled regions. These equations are particularized and solved for several axisymmetric membranes which are subjected to various types of boundary conditions and external forces. Two different types of strain energies are considered, i.e. Harmonic material which is used to obtain analytical solutions for the problem of a twisted annulus, and Ogden's material which is used to solve the problems with pole-type singularities caused by point-load.

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CHAPTER 1. TENSION FIELD THEORY

1.1 Introduction

The work summarized in this thesis is a study on the nonlinear wrinkling of elastic membranes. Several axisymmetric problems were solved using tension field theory.

Wrinkling is a common phenomenon associated with certain deformations of elastic membranes. A classical example illustrating this special feature of membrane structures was presented by Wagner [1.1] in analyzing a strip of an extremely thin metal sheet supported perpendicular to the plane of the sheet and acted upon by uniform planar shear load along the edges. He showed that, up to a certain intensity of the shear load, a corresponding uniform plane state of stress is produced in the sheet. If the shear load is increased beyond a certain intensity, buckling occurs. However if the distance of the opposite edges of the sheet is kept constant, then the boundary shear load can approach an intensity many times that of the initial buckling load without collapse of the structure as such. The structure sustains such a deformation due to the formation of wrinkles in the sheet. In order to analyze the loading capacity of wrinkled membrane structures, it is necessary to study the stress and strain fields of the concerned membrane structures. Our emphasis is focused on the stress-strain analysis in the wrinkled parts of membranes as it is finitely deformed.

Once the wrinkles are present, the basic governing equations in the wrinkled region must be modified, and this necessitates the reformulation of the classical membrane theory in an essential way. To this end, certain assumptions must be made in relation to the behaviours of wrinkled membranes. It is commonly assumed that the direction of

a wrinkle coincides with the direction of a simple tension. The stress resultant along a wrinkle is assumed to be tensile, while the transverse stress resultant is zero. The uniaxial stress field defined in this way forms a tension field in the wrinkled region. By definition, a tension field should only be uniaxial in the sense that it should be the stress field with only a single principal component. Thus, this definition is valid for any curved surface and no restriction is made on the severity of deformation, i.e. it is valid for both infinitesimal and finite deformations.

Most of the previous investigations on tension field were accomplished by means of infinitesimal displacement theory. Wagner [1.1] was the first researcher to conceive the concept of tension field in his search for the deformation of the wrinkled thin sheet subjected to edge shear loads. The basic idea is that wrinkles are treated as being spaced infinitesimally close together and no transverse stress is exerted across a wrinkle. This defines a two-dimensional plane tension field. The deformation obtained gives the mean shape of the deformed surface, not including the detailed configuration of the wrinkles. Reissner [1.2] later extended Wagner's theory by introducing an artificial curvilinear anisotropy into the homogeneous membrane material and the curvilinear anisotropy was assumed to conform with the principal stress direction. In that setting, a separate elastic modulus was associated with each principal direction. The wrinkling is thus indicated by the diminution of one of the moduli to zero. An illustrative example used by Reissner to demonstrate his theory was the an annular membrane which was axisymmetrically deformed through the rotation of a concentric rigid hub attached to the annulus's inner boundary.

For isotropic membranes, Mansfield [1.3] gave yet another approach via the use of an energy principle. He had shown that the real distribution of wrinkles maximizes the equilibrium strain energy density. His theory was based on the theorem that the lines of tension in a wrinkled membrane are exactly in a direction for which the strain energy is at a maximum. Stein and Hedgepeth [1.4] introduced the concept of a variable Poisson's ratio with particular reference to the region that is partly wrinkled. The formulation of the problem can be regarded as a modification of Reissner's theory. Again, the problem of the annular membrane was considered, and the theoretical approximations were in good agreement with experimental data. Later on, by neglecting the crests and troughs of wrinkles, Zak [1.5] considered the form of the equilibrium equations for a "pseudo-surface" in a coordinate system aligned with the directions of the principal stress resultants. Using the assumption that the compressive stresses are zero, Zak had shown that, in the absence of external surface loads, the crest of the wrinkles form a family of straight lines, i.e. the pseudo-surface is ruled.

The first non-linear treatment of wrinkling was due to Wu [1.6], who presented a comprehensive analysis of wrinkling on finitely deformed membranes. The formulation involved a rather complicated mathematical model to describe the degree of wrinklyness. By avoiding the detailed structure of wrinkles, the wrinkled surface is replaced by a smoothed-out pseudo surface which represents the mean shape of the wrinkled surface. This is analogous to Zak's treatment. One then requires that not only must this pseudo-surface be in equilibrium, but on it, the compressive stress resultant must be zero. Obviously, the stretch of the pseudo-surface in the direction of the zero principal stress

resultant will not be equal to that of the actual wrinkled surface. To describe such differences, a kinematic variable called wrinkle-strain was introduced to measure the physical wrinklyness of the actual membrane. The value of this parameter was determined by the condition that the stress perpendicular to the wrinkle line is zero. The introduction of wrinkle-strain is justified by the fact that the vanishing of one compressive principal stress yields an additional degree of freedom to satisfy the requirement of strain compatibility.

The significant progress made in wrinkling analysis for isotropic membranes was due to Pipkin [1.7]. Determination of a stable equilibrium deformation of the membrane subjected to conservative forces can be posed as a minimization problem of the total potential energy. As the necessary conditions for stable equilibrium deformations, Pipkin [1.7] presented a set of constraint conditions on the derivatives of the strain energy functions. Independently, Steigmann [1.8] obtained a set of equivalent necessary conditions which state that a stable or neutrally stable potential energy minimizer exists only if the principal stresses in the deformed membrane are pointwisely non-negative. However, the state of stress associated for certain equilibrium deformations will usually violate at least one of these conditions. These situations are encountered when the real physical membrane exhibits wrinkles. Under this circumstance, the classical membrane theory yields either no stable solutions or no solution at all.

An approach to overcome this difficulty was provided by Ball and James [1.9] and James [1.10] who demonstrated that energy-minimizing sequences of deformation are characterized by finely spaced discontinuities in the corresponding deformation gradients.

The deformation sequences converge to a deformation whose gradient itself is not a minimizer of the strain energy. Pipkin [1.7] demonstrated that the minimum of the potential energy, which can not be attained by solutions of classical membrane theory, could be achieved in a smooth limit of a sequence of deformations involving finely spaced wrinkles. The strain energy associated with the sequences of deformation is called relaxed strain energy density which represents the average strain energy of the regions containing continuously distributed wrinkles. Steigmann [1.12] later expanded this concept to the arbitrarily curved elastic membranes and incorporated tension field theory into the classical membrane theory. It is accomplished simply by replacing the ordinary strain energy function by a suitable relaxed strain energy density, and the theory accounts for the possible wrinkling automatically. The new version of membrane theory has prevailing advantages in the following aspects: (1) Wrinkling analysis is integrated into the ordinary membrane theory simply by using the relaxed strain energy density. (2) The introduction of a kinematical parameter called Natural width the value of which identifies the boundary separating the tense and wrinkled regions. It is due to this variable that makes the numerical computations possible.

1.2 Kinematics, Isotropic membranes and potential energy

Within the context of direct surface theory, a membrane is defined as a two-dimensional flexible continuum comprising of a collection of material particles situated on the surface embedded in a three-dimensional Euclidean space. When the membrane is deformed, the position of the material particles as well as the shape of the surface vary. If the material

is elastic, the membrane is called an elastic surface. The two-dimensional elastic surface theory has been established to reflect the main features of elastical thin shells.

In general, there are two entirely different approaches that we may take to construct the two-dimensional surface theory characterizing certain motions and material responses which are confined to two-dimensional regions. In the first approach [1.14], governing equations are obtained as an approximation to the governing equations of fully three-dimensional shell theory. In this formulation, the membrane theory is derived as a limiting case of the shell theory which occurs when the thickness of the shell becomes considerably small. The second approach [1.15], however, is usually called *direct theory* in which a membrane is regarded as a two-dimensional elastic surface and the governing equations are derived without reference to the three-dimensional theory. Various discussions on this method can be found in [1.16], [1.17] and [1.12].

For the problems under consideration, we follow the approach of *direct theory*. In what follows, we give a brief review of some important results of the surface theory to which will be frequently referred in the subsequent analysis. Although much of the basic materials have been discussed in various forms elsewhere [1.18,19,20,21,22], its inclusion here serves not only to establish the relevant notations, but also provides a basis for later reference.

We adopt the convention that a vector or a tensor is denoted by a letter with an underbar. Index notation is a means of labelling the elements of a vector or tensor with respect to a basis to which they are referred. An index is a lowcase letter which appears as either a subscript or a superscript. In the meantime, Latin index represents any of the

numbers 1, 2 or 3, whereas Greek index takes on the number 1 or 2. Summation convention is defined as follows, any term in which an index is repeated denotes a sum of all terms obtained by assigning numbers to either the subscript or superscript.

In a three-dimensional Euclidean space E_3, a fixed global *Cartesian coordinate system* with origin O is employed with respect to which the places of points in E_3 are defined. The Cartesian coordinates and the base vectors are denoted by a triplet $\{x_i, i=1,2,3\}$ and a set of three orthonormal vectors $\{\underline{e}_i, i=1,2,3\}$, respectively.

Let a fixed surface S with the boundary ∂S in E_3 represent a membrane in its *reference state* on which the material particles are identified by a system of *convected curvilinear coordinates*. It assigns each material particle on S a pair of coordinates (θ^1, θ^2). In the terminology of elasticity theory, (θ^1, θ^2) are called Lagrangian coordinates [1.18] which remain a one-to-one correspondence with the material particles throughout the deformation.

In E_3, the position of the point p on S is determined by

$$\underline{x} = \underline{x}(\theta^\alpha) = x_i(\theta^\alpha)\underline{e}_i, \qquad \alpha = 1, 2; \ i=1,2,3 \tag{1.2.1}$$

where $\underline{x}$ is the position vector which extends from the origin O to the point p on the surface S. $\{x_i\}$ are the Cartesian components of $\underline{x}$ relative to the basis $\{\underline{e}_i\}$. Curves along which either one of (θ^1,θ^2) remains constant are called *convected coordinate curves* on S [1.18] (see Fig.1.2.1). For example, θ^1-curves are defined as a family of curves along each one of which θ^2 = const., different points on θ^1-curve are then distinguished by values of θ^1. We further assume that $\underline{x}$ is continuously differentiable with respect to θ^α as many times as it is required except at certain points or across certain curves on S.

The natural base vectors $\underline{A}_\alpha$ induced by θ^α on S are defined by

$$\underline{A}_\alpha = \underline{x}_{,\alpha}, \qquad \alpha = 1, 2 \tag{1.2.2}$$

where $(\,)_{,\alpha}$ denote the covariant derivatives with respect to θ^α. By definition, $\underline{A}_\alpha$ are not necessarily unit vectors. Equations (1.2.2) shows that $\underline{A}_\alpha$ are tangents to the θ^α-curves at point p on S as shown in Fig.1.2.1. Thus $\underline{A}_\alpha$ span the tangent plane of S at p.

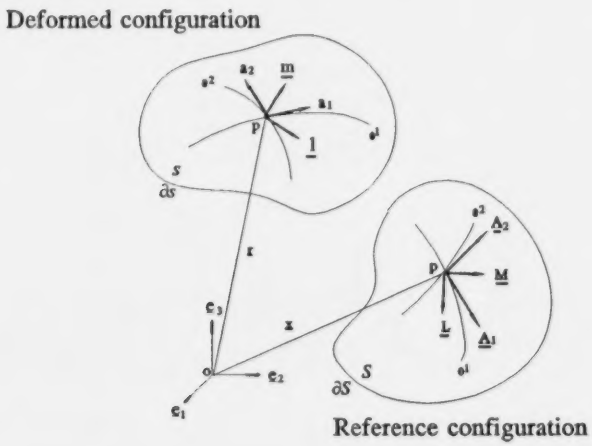

Fig. 1.2.1 Reference and deformed surfaces.

An infinitesimal vector $d\underline{x}$ connecting two adjacent material particles with coordinates θ^α and $\theta^\alpha + d\theta^\alpha$ on the reference surface S is defined by the equation:

$$d\,\underline{x} = \frac{\partial \underline{x}}{\partial \theta^\alpha}\, d\theta^\alpha = \underline{A}_\alpha\, d\theta^\alpha \tag{1.2.3}$$

in which relation (1.2.2) has been used. In E_3, the length of $d\underline{x}$, denoted by dS, is defined as the square root of the scalar product of $d\underline{x}$ with itself [1.19]. From (1.2.3),

we have

$$(ds)^2 = d\underline{x} \cdot d\underline{x} = A_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta \tag{1.2.4}$$

where $A_{\alpha\beta} = \underline{A}_\alpha \cdot \underline{A}_\beta$ and the entity of which is called metric tensor of the reference surface S [1.19]. The quadratic form in $d\theta^\alpha$ is called the first fundamental form of surface. Clearly, metric tensor is symmetric.

The unit normal to S at point p is denoted by $\underline{N}$. It is oriented in such a way that $\underline{A}_1$, $\underline{A}_2$ and $\underline{N}$, in this order, form a right-handed system. It then follows that

$$\underline{N} = A^{-1/2}(\underline{A}_1 \times \underline{A}_2) \tag{1.2.5}$$

provided $A \neq 0$. Here, $A = \det(A_{\alpha\beta}) = |\underline{A}_1 \times \underline{A}_2|^2$ according to Gurtin [1.20]. det( ) is the determinant of the matrix.

The metric tensor can be used to derive the reciprocal base vectors $\underline{A}^\alpha$ which are defined as vectors satisfying $\underline{A}^\alpha \cdot \underline{A}_\beta = \delta^\alpha{}_\beta$, where $\delta^\alpha{}_\beta$ is the two-dimensional Kroneckel delta. From equation (1.2.4), the reciprocal metric components are uniquely defined by $(A^{\alpha\beta}) = (A_{\alpha\beta})^{-1}$ where the superscript -1 denotes the matrix inverse. Thus, $\underline{A}^\alpha$ can be calculated by the relation

$$\underline{A}^\alpha = A^{\alpha\beta}\underline{A}_\beta \tag{1.2.6}$$

We consider a deformation which carries the membrane from its reference surface S to a new surface called s. As a result, the point p with coordinates θ^α on S is displaced from the original place $\underline{x}$ to the new place $\underline{r}$. If $\underline{r}$ is referred to the same origin O as $\underline{x}$ was, it may be expressed in the form

$$\underline{r} = \underline{r}(\underline{x}) = r_i(\theta^\alpha)\underline{e}_i \tag{1.2.7}$$

where r_i are the Cartesian components of $\underline{r}$ with respect to the global basis $\{\underline{e}_i\}$. We

assume that $r_i(\cdot)$ are piecewisely differentiable with respect to θ^α. Using lowcase letters to denote the corresponding quantities on s, similar kinematic relations to those in equations (1.2.2, 3, 4) can be obtained as follows

$$\underline{a}_\alpha = \frac{\partial \underline{r}}{\partial \theta^\alpha}; \qquad d\,\underline{r} = \underline{a}_\alpha\, d\theta^\alpha; \qquad a_{\alpha\beta} = \underline{a}_\alpha \cdot \underline{a}_\beta; \tag{1.2.8}$$
$$(ds)^2 = a_{\alpha\beta}\, d\theta^1\, d\theta^2; \qquad a = \det(a_{\alpha\beta}).$$

where $\underline{a}_\alpha$ are the vectors into which the base vector $\underline{A}_\alpha$ are deformed and $\underline{a}_\alpha$ are tangent to the deformed θ^α-curves. $a_{\alpha\beta}$ are the covariant components of the metric tensor of the deformed surface s.

The deformation of the differential material vector $d\underline{x}$ defined by (1.2.3) can be determined by the *deformation gradient* $\underline{F}$ [1.20]. It is a linear transformation defined by the relation

$$d\underline{r} = \underline{F}\, d\underline{x} = \underline{\nabla} \underline{r}\, d\underline{x} \tag{1.2.9}$$

provided that $\underline{r}$ has piecewise C^1 continuity. The notation $\underline{\nabla} = \mathrm{Grad}(\,) = (\,)_{,\alpha}\, \underline{A}^\alpha$ is the gradient operator on S. The local nature of the deformation is therefore embodied in the properties of $\underline{F}$. Differential vectors $d\underline{x}$ at $\underline{x}$ and $d\underline{r}$ at $\underline{r}$ appearing in (1.2.9) represent the same material line element on the reference and deformed surfaces, respectively.

Substituting (1.2.3) and $(1.2.8)_2$ into (1.2.9), $\underline{F}$ can be expressed as

$$\underline{F} = \underline{a}_\alpha \otimes \underline{A}^\alpha. \tag{1.2.10}$$

which, according to an extension of the polar decomposition theorem presented by Pipkin [1.7] and Chadwick [1.21], can be written in the form

$$\underline{F} = \lambda\, \underline{l} \otimes \underline{L} + \mu\, \underline{m} \otimes \underline{M}, \tag{1.2.11}$$

where λ, μ are the real principal stretches. $\{\underline{l},\underline{m}\}$ and $\{\underline{L},\underline{M}\}$ are the orthonormal principal vectors spanning the tangent planes of the deformed and reference surfaces. Principal stretch λ (or μ) is defined as the extensional ratio of an infinitesimal material line element with length ds in the direction of $\underline{l}$ (or $\underline{m}$) on s to the corresponding material line element which had the length dS in the direction of $\underline{L}$ (or $\underline{M}$) on S.

The right Cauchy-Green strain tensor $\underline{C}$ is defined as

$$\underline{C} = \underline{F}^T\underline{F} \tag{1.2.12}$$

which, upon the substitution of equation (1.2.11) into (1.2.12), has the spectral form

$$\underline{C} = \lambda^2\, \underline{L} \otimes \underline{L} + \mu^2\, \underline{M} \otimes \underline{M}. \tag{1.2.13}$$

The principal invariants of $\underline{C}$ are thus computed from (1.2.13) as

$$\det(\underline{C}) = J^2, \quad \mathrm{tr}(\underline{C}) = I^2 - 2J, \tag{1.2.14}$$

in which

$$I = \lambda + \mu, \quad J = \lambda\,\mu. \tag{1.2.15}$$

where $\mathrm{tr}(\underline{C})$ and $\det(\underline{C})$ are the trace and determinant of $\underline{C}$. J measures the deformed surface area per unit of reference surface.

We suppose that the membrane is made of hyperelastic material. The responses of a hyperelastic material to a deformation are characterized by a strain energy function $W(\underline{F})$ measured per unit area of the reference surface. An elastic membrane is called hyperelastic if the Piola stress tensor $\underline{T}$ [1.12] is the derivative of $W(\cdot)$ with respect to the deformation gradient $\underline{F}$. If $\underline{F}$ is known, the stresses at each material particle on the deformed surface are completely determined by the constitutive relation. Strain energy defined in this way depends only on the relative change in distance between neighbouring

material particles. Effects of changes in the local curvatures of the reference and deformed surfaces are not taken into account. Thus, $W(\underline{F})$ does not consider bending resistance commonly associated with elastic thin shells in three-dimensional theories.

By definition, Piola stress tensor $\underline{T}$ induced by $\underline{F}$ is defined as

$$\underline{T} = \frac{\partial W(\underline{F})}{\partial \underline{F}} = \underline{T}^{\alpha} \otimes \underline{A}_{\alpha}$$
$$\text{and} \quad \underline{T}^{\alpha} = \frac{\partial W(\underline{F})}{\partial r^{i}{}_{,\alpha}} \underline{e}_{i} \tag{1.2.16}$$

where $\underline{T}^{\alpha}$ is usually referred to as the stress vector. The force transmitted across a material arc $d\underline{x}$, with length $ds = |d\underline{x}|$ and unit normal vector $\underline{\nu}$, is $\underline{t}\, ds$, where

$$\underline{t} = \underline{T}\, \underline{\nu} \tag{1.2.17}$$

where $\underline{\nu}$ is oriented so that $\underline{\nu}\, ds = d\underline{x} \times \underline{N}$.

According to [1.12], strain energy functions for isotropic membrane depend on $\underline{C}$ through its invariants. In view of relations (1.2.15), we can write

$$W(\underline{F}) = w(\lambda,\mu) \tag{1.2.18}$$

where $w(\cdot)$ is a symmetric function of λ and μ in the sense that $w(\lambda,\mu) = w(\mu,\lambda)$.

By means of (1.2.18), Piola stress tensor $\underline{T}$ can then be expressed in terms of the principal vectors $\{\underline{L},\underline{M}\}$, $\{\underline{l},\underline{m}\}$ in the form

$$\underline{T} = w_{\lambda}\, \underline{l} \otimes \underline{L} + w_{\mu}\, \underline{m} \otimes \underline{M} \tag{1.2.19}$$

where w_{λ} and w_{μ} are partial derivatives of $w(\cdot)$ with respect to λ and μ respectively.

For the problems under consideration, the external forces applied on the membrane are assumed to be dead forces. We consider three different types of these forces. One is the uniform internal pressure of prescribed intensity, the second one is the prescribed

dead force applied along a certain part of the boundary ∂S, and finally the point-loads of prescribed intensity and direction.

In [1.23], Steigmann derived the potential function for the internal pressure with the prescribed intensity p^d, where superscript ^{d} refers to the dead force. In what follows, we will use ^{d} to denote all dead forces. If the deformed membrane is a closed surface subjected to p^d, the potential function $P_p[\underline{r}]$ is

$$P_p[\underline{r}] = \frac{p^d}{3} \int\int_S \underline{r} \cdot \underline{n} \, J \, dA \tag{1.2.20}$$

where dA is the elemental area of the reference surface S. $\underline{r}$ is the position vector on the deformed surface s with the unit normal $\underline{n}$ in the direction of the pressure. J is defined in $(1.2.15)_2$.

The potential function $P_b[\underline{r}]$ for the force $\underline{F}^d$ prescribed along ∂S_t is

$$P_b[\underline{r}] = \int_{\partial S_t} \underline{F}^d \cdot \underline{r} \, ds \tag{1.2.21}$$

where $\partial S_t \in \partial S$. $\underline{F}^d$ may be regarded as an assigned function of position per unit initial arc length on ∂S_t, i.e. $\underline{F}^d = \underline{F}^d(S)$, for $S \in \partial S_t$.

The membrane may also be subjected to a concentred force of prescribed intensity $\underline{F}_c^{\,d}$, the potential function $P_c[\underline{r}_c]$ is simply a scalar product

$$P_c[\underline{r}_c] = \underline{F}_c^{\,d} \cdot \underline{r}_c \tag{1.2.22}$$

where $\underline{r}_c$ is the position at which $\underline{F}_c^{\,d}$ is applied.

The total potential energy $E[\underline{r}]$ is defined as the total stored strain energy stored in the membrane minus the total potential energies of the external forces. In view of (1.2.20,

21, 22), $E[\underline{r}]$ is written as

$$E[\underline{r}]=\int\int_S H(\underline{F};\underline{r})\,dA \;-\; P[\underline{r}] \tag{1.2.23}$$

where

$$H(\underline{r}) \;=\; W(\underline{F}) - \frac{1}{3}p^d J\underline{n}\cdot\underline{r}\;; \qquad P[\underline{r}]=\int_{\partial S_t}\underline{F}^d\cdot\underline{r}\,ds \;+\; \underline{F}^d{}_c\cdot\underline{r}_c \tag{1.2.24}$$

1.3 Energy minimizers and the Relaxed strain energy

For the membrane subjected to conservative forces, stable equilibrium deformations are defined to be the local minimizers of the total potential energy [1.7]. We can regard the problem of finding a stable equilibrium deformation as a minimization problem of the total potential energy $E[\underline{r}]$ [1.7], [1.12]. The principle of minimum potential energy can be stated as follows [1.24]: among all kinematically admissible variations $\delta\underline{r}$, the actual ones are those that render the total potential energy stationary. This is

$$\delta\, E[\underline{r}] \;=\; 0 \tag{1.3.1}$$

for an equilibrium $\underline{r}$. δ denotes the variation operation. In view of (1.2.23, 24), $E[\underline{r}]$ is the functional of $\underline{r}$ which is regarded as the quantity subject to variation. The variation, denoted by $\delta\underline{r}$, should be chosen such that $\delta\underline{r}$ does not violate the prescribed kinematic boundary conditions.

Following Steigmann [1.12] and (1.3.1), the equations of equilibrium and the natural boundary conditions are

$$\begin{aligned} &A^{-\frac{1}{2}}(A^{\frac{1}{2}}\underline{T}^\alpha)_{,\alpha} \;+\; p^d J\underline{n} \;=\; 0; \quad \underline{x}\in S \\ &\underline{T}\,\underline{\nu}=\underline{F}^d, \quad \underline{x}\in\partial S_t \end{aligned} \tag{1.3.2}$$

On the remaining boundary $\partial S_b = \partial S - \partial S_t$, the deformation is prescribed, i.e.

$$\underline{r} = \underline{r}_b, \quad \underline{x} \in \partial S_b. \tag{1.3.3}$$

where $\underline{T}$ and $\underline{T}^\alpha$ are the respective Piola stress tensor and the stress vector defined in (1.2.16). A is the determinant of the metric tensor used in (1.2.5). $\underline{\nu}$ is the outward unit normal to the boundary ∂S_t.

As a common practice in the application of minimum potential energy principle, the above discussion should naturally lead to the questions: Is the solution obtained from (1.3.2) stable?, Or is there a stable solution at all that minimizes the potential energy (1.2.23). In continuum mechanics, there are several stability criteria which lead to distinct stability theories. In this work, we shall consider the energy criterion (mechanical) for determining the stability of equilibrium configurations of elastic membranes subjected to conservative forces.

The essential concept of the energy criterion can be stated as follows: let $\underline{r}$ and $\delta\underline{r}$ denote the equilibrium configuration and the perturbation, the total potential energy corresponding to the perturbation $\delta\underline{r}$ can be expressed as

$$E[\underline{r}+\delta\underline{r}] = E[\underline{r}] + \delta E[\underline{r}] + \delta^2 E[\underline{r}] + \delta^3 E[\underline{r}] + \ldots \tag{1.3.4}$$

where $\delta E[\underline{r}]$, $\delta^2 E[\underline{r}]$,..., are the first, second,..., variations of $E[\cdot]$ at $\underline{r}$. They are linear, quadratic,..., with respect to $\delta\underline{r}$. Since $\underline{r}$ is in equilibrium, it is necessary that $\underline{r}$ satisfies (1.3.1). With these preliminaries, neglecting terms of higher order, we may now conclude that the stability of the original deformation $\underline{r}$ in its neighbourhood can be determined by the sign of the second variation $\delta^2 E[\underline{r}]$: the configuration is stable if $\delta^2 E[\underline{r}] > 0$ for all kinematically admissible perturbations. Whereas, the configuration is

unstable if $\delta^2 E[\underline{r}] < 0$ for at least one kinematically admissible perturbation. Stability of $\underline{r}$ implies that $\underline{r}$ is a weak relative minimizer of $E[\cdot]$ at $\underline{r}$.

Following the energy criterion, Steigmann [1.8] concluded that the stable or neutrally stable equilibrium configuration of an elastic membrane subjected to conservative forces must have a stress field in which the principal stresses are everywhere non-negative. These results were used in classical membrane theories as the primitive ad hoc assumptions.

If the membrane is free of external forces, i.e. $p^d = 0$, $\underline{F}^d = 0$ and $\underline{F}^d_c = 0$, the potential energy (1.2.23) becomes a single integral $E[\underline{r}] = \iint_S W(\underline{F})dA$. According to Graves's theorem [1.29], a deformation $\underline{r}$ minimizes $E[\underline{r}]$ with respect to bounded, continuous perturbations $\underline{u}$ only if the strain energy $W(\underline{F})$ is rank-one convex at the gradient $\underline{F}$ at every point in S. This necessary condition is stated as an inequality

$$W(\underline{F} + \underline{u} \otimes \underline{n}) - W(\underline{F}) - \underline{u} \cdot \underline{T}\underline{n} \geq 0 \qquad (1.3.5)$$

which must be satisfied for every perturbation $\underline{u} \in R^3$, $\underline{n} \in R^2$. $\underline{T}$ is the Piola stress tensor.

Steigmann [1.8] has demonstrated that inequality (1.3.5) is satisfied identically if $W(\underline{F})$ is replaced by the loading potential $P_p[\underline{r}]$ defined by (1.2.20). Thus (1.3.5) also holds when $W(\cdot)$ is replaced by $H(\cdot)$ defined by $(1.2.24)_1$. In addition, if the membrane is also subjected to distributed forces along the boundary ∂S and point-loads at certain points in the interior of S, perturbations may be performed by holding the entire boundary ∂S and the points of the application of the point-loads fixed during the perturbation. As a result, the work done in the course of the perturbation by these forces

are zero. Hence, (1.3.5) is still valid as a necessary condition for the equilibrium deformations that minimize E[$\underline{r}$].

Rank-one convexity of W(·) at $\underline{F}$ ensures that the second derivative of W($\underline{F}$ + θ $\underline{u} \otimes \underline{n}$) with respect to θ is non-negative for all $\underline{F}$, $\underline{u}$ and $\underline{n}$. Using the notation

$$\frac{\partial^2 W}{\partial F_{ia} \partial F_{jb}} = W_{iajb}, \tag{1.3.6}$$

the so-called *Legendre-Hadamard inequality*, a consequence of (1.3.5), is

$$W_{iajb} u_i n_a u_j n_b \geq 0 \tag{1.3.7}$$

For isotropic materials, Pipkin [1.7] used (1.2.18) in (1.3.7) to form a set of restrictions on the derivatives of $w(\lambda,\mu)$. These are the necessary and sufficient conditions for Legendre-Hadamard inequality. These are

$$w_\lambda \geq 0, \quad w_\mu \geq 0, \quad w_{\lambda\lambda} \geq 0, \quad w_{\mu\mu} \geq 0, \quad a \geq 0 \tag{1.3.8}$$

$$(w_{\lambda\lambda}\, w_{\mu\mu})^{1/2} - w_{\lambda\mu} \geq b - a, \quad (w_{\lambda\lambda}\, w_{\mu\mu})^{1/2} + w_{\lambda\mu} \geq -b - a, \tag{1.3.9}$$

where

$$a = (\lambda w_\lambda - \mu w_\mu)/(\lambda^2 - \mu^2) \quad \text{and} \quad b = (\mu w_\lambda - \lambda w_\mu)/(\lambda^2 - \mu^2). \tag{1.3.10}$$

These conditions are obtained by selecting $\underline{u}$ and $\underline{n}$ independently [1.3]. Inequalities in (1.3.8, 9, 10) must be satisfied by any equilibrium deformation that minimizes E[$\underline{r}$].

Under certain circumstances, at least one of (1.3.8) is violated for some deformations. To show this, let the undeformed membrane be a unit square sheet, stretched and held in equilibrium by forces in the plane of the sheet. The forces are applied uniformly and normally to the edges. The lateral surface of the sheet is assumed to be traction free ($p^d = 0$).

A homogeneously deformation in a (x_1,x_2)-plane can be described by the equation

$$\underline{r} = \underline{F}\underline{x}; \tag{1.3.11}$$

where $\underline{F}$ is a constant tensor. Deformation (1.3.11) automatically satisfies the equilibrium equations (1.3.2).

The deformation gradient $\underline{F}$ is simply

$$\underline{F} = \lambda_\alpha \underline{e}_\alpha \otimes \underline{e}_\alpha \tag{1.3.12}$$

where $\{\underline{e}_\alpha, \alpha=1,2\}$ are the orthonormal. $\lambda_1 = \lambda$, $\lambda_2 = \mu$ are the principal stretches in the direction of $\underline{e}_1$ and $\underline{e}_2$, respectively. As a result of the deformation (1.3.11), an initial unit square is deformed into a rectangle of dimensions λ by μ. For definiteness, we assume $\lambda > \mu$. The deformation is symmetric about x_1- and x_2-axis simultaneously, therefore, principal vectors are such that $\underline{l} = \underline{L} = \underline{e}_1$, $\underline{m} = \underline{M} = \underline{e}_2$. Using these facts in (1.2.19), the Piola stress tensor is

$$\underline{T} = w_\lambda \underline{e}_1 \otimes \underline{e}_1 + w_\mu \underline{e}_2 \otimes \underline{e}_2 \tag{1.3.13}$$

By setting $\underline{v} = \underline{e}_1$ and $\underline{v} = \underline{e}_2$ in (1.2.17) independently, the stress resultants needed on the edges to maintain the equilibrium are

$$\underline{t}_1 = w_\lambda \underline{e}_1; \quad \underline{t}_2 = w_\mu \underline{e}_2 \tag{1.3.14}$$

It is observed from (1.3.14) that $\underline{t}_1$ and $\underline{t}_2$ must be tensile stresses in order for w_λ and w_μ to be positive to satisfy $(1.3.8)_{1,2}$. When either of them, say $\underline{t}_2$, is compression, $(1.3.14)_2$ yields $w_\mu < 0$. When this happens, $(1.3.8)_2$ is violated and, consequently, the deformation is unstable.

For the unit square sheet under consideration, the least stable state of equilibrium is such that $\underline{t}_1 > 0$, $\underline{t}_2 = \underline{0}$ so that $(1.3.10)_2$ is satisfied as an identity. This corresponds

to a state of uniaxial tension in the deformed state. In such a state, μ is less than unity (<1). We record the value of μ for which $(1.3.8)_2$ is an equality, i.e. $w_\mu(\lambda,\cdot) = 0$, and denote this μ by $v(\lambda)$. If $w(\cdot)$ satisfies $(1.3.8)_4$ strictly, $w_\mu(\lambda,\cdot) > 0$ for $\mu > v(\lambda)$, and $w_\mu(\lambda,\cdot) < 0$ when $\mu < v(\lambda)$. Since $w(\lambda,\mu)$ is symmetric with respect to λ and μ, then $w_\lambda(\cdot,\mu) > 0$ for $\lambda > v(\mu)$, and $w_\lambda(\cdot,\mu) < 0$ when $\lambda < v(\mu)$. It is important that $v(\lambda)$ varies with λ and is independent of the reference area. Thus, it is a material property. e.g. if the material is isotropic and incompressible, $v(\lambda) = \lambda^{-1/2}$ [1.7].

With the aid of $v(\lambda)$, inequalities $(1.3.8)_{1,2}$ define the ranges of λ and μ for stable states of equilibrium. They are

$$\mu > v(\lambda); \quad \lambda > v(\mu) \tag{1.3.15}$$

In the terminology of Pipkin [1.7], $v(\lambda)$ is defined as the *Natural width* in simple tension.

Usually, under certain circumstances, an equilibrium solution of Euler-Lagrange equations for a certain boundary conditions furnishes principal stretches that violate at least one of the conditions (1.3.15) on some parts of the deformed surface. This fact may be exemplified by the deformation in which part of the membrane is wrinkled. Occurrence of wrinkles is due to the presence of compressive stresses induced by the deformation. When such a state of compressive stress is encountered, either $(1.3.8)_1$ or $(1.3.8)_2$ is violated. Therefore, the energy minimization problem which is formulated based on the ordinary strain energy $w(\lambda,\mu)$ is inadequate to account for this type of deformations. Therefore, no stable solutions can be found.

Because we are considering the membrane whose strain energy $w(\lambda,\mu)$ is indifferent to

the changes in local curvatures of the surface, consequently, one is allowed to fold such a membrane along any line on the deformed surface without changing its strain energy value. Using this characteristics to consider membranes which are wrinkled, the stretch in the direction of wrinkling may be achieved by a process of continuous folding to the actual value. Since folding does not alter the original state of stretches, the strain energy $w(\lambda,\mu)$ associated with the folding is the same as that evaluated before the folding.

To illustrate the process of folding, we consider the deformation of a unit square sheet. Let the unit square be deformed into a rectangle $\Re$ of dimensions $\lambda > 1$ and $\mu = v(\lambda)$ by a simple tension. According to our constitutive hypothesis, the state of stress is an uniaxial tension with $w_\lambda > 0$ and $w_\mu = 0$. To consider the subsequent folding operations, we employ a system of Cartesian coordinates (x,y,z) oriented in such a way that base vectors $\underline{i}$, $\underline{j}$ associated with coordinates x, y are parallel to λ, μ directions respectively. The normal vector is $\underline{k} = \underline{i} \times \underline{j}$.

Let the rectangle $\Re$ be divided into N strips parallel to the tensile axis (λ-direction), each strip has the width $v(\lambda)/N$. We then fold $\Re$ along the boundaries of the strips. After folding, each fold retains its original width but is rigidly displaced and inclined at an angle α to its original plane. A cross-section of the folded $\Re$, denoted by $\mathscr{E}$, is shown in Fig.1.3.1a.

Let $\underline{m}_N$ be the unit vector defined by

$$\underline{m}_N = \cos\alpha\, \underline{j} + s(y) \sin\alpha\, \underline{k}, \quad 0 < \alpha < \pi/2 \tag{1.3.16}$$

where the sub-N denotes the N^{th} member of the folding sequence in which $v(\lambda)$ is divided into N strips. α is the angle between $\underline{m}_N$ and $\underline{j}$ which α is positive if it is measured

counterclockwisely from $\underline{j}$. The value of $|\alpha|$ is assumed to be a constant for each strip.

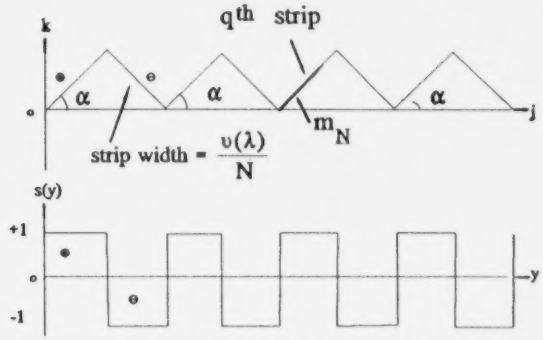

Fig.1.3.1 Folding process

s(y) is the sign function defined by the relation

$$s(y)=\begin{cases} +1 & y \in \oplus \text{ region} \\ -1 & y \in \ominus \text{ region} \end{cases} \qquad (1.3.17)$$

According to Fig. 1.3.1, a point (x,y) lying on the right edge of q^{th} strip in the original rectangle is displaced by the N^{th} folding sequence to the new place defined by

$$\underline{r}_N(x,y) = x\lambda\,\underline{i} + \sum_{i=1}^{q}\left(\frac{i}{N}\right)\nu(\lambda)\cos\alpha\,\underline{j} + \sum_{i=1}^{q}\left(\frac{i}{N}\right)\nu(\lambda)\sin\alpha\,s(y)\,\underline{k} \qquad (1.3.18)_a$$

or, in terms of $\underline{m}_N$ defined in equation (1.3.16), by

$$\underline{r}(x,y)_N = x\lambda\,\underline{i} + y\,\nu(\lambda)\,\underline{m}_N \qquad (1.3.18)_b$$

where $y = q/N \in [0, 1]$. By examining the last term in $(1.3.18)_a$, the vertical displacements of the edges of a strip are either $\upsilon(\lambda)/N \sin\alpha$ or zero.

From $(1.3.18)_b$, the deformation gradient $\underline{F}_N$ associated with N^{th} folding sequence is

$$\underline{F}_N = \lambda\underline{i}\otimes\underline{i} + \nu(\lambda)\underline{m}_N\otimes\underline{j} \qquad (1.3.19)$$

As the number of strips approaches infinity ($N \to \infty$), strip width $\upsilon(\lambda)/N$ approaches

zero. As a result, the last term in (1.3.18)$_4$ vanishes due to the presence of sign-alternating function s(y). This reduces the amplitude of the folds to zero, and the sequence $\underline{r}_N$ converges to a smooth differentiable deformation

$$\tilde{\underline{r}} = x\,\lambda\,\underline{i} + y\,\nu(\lambda)\cos\alpha\,\underline{j} \tag{1.3.20}$$

which has the deformation gradient

$$\tilde{\underline{F}} = \lambda\,\underline{i}\otimes\underline{i} + \tilde{\mu}\,\underline{j}\otimes\underline{j} \tag{1.3.21}$$

where $\tilde{\mu} = \nu(\lambda)\cos\alpha$ that may vary arbitrarily in the range $[0, \nu(\lambda)]$ in response to the value of α in the interval $(0, \pi/2]$. Obviously, $\tilde{\underline{F}}$ furnishes a principal stretch that is smaller than the natural width $\nu(\lambda)$ in simple tension.

Although the sequence of deformation $\underline{r}_N$ (1.3.18) converges to the smooth surface $\tilde{\underline{r}}$ (1.3.20), its gradient $\underline{F}_N$ (1.3.19) does not converge to $\tilde{\underline{F}}$ (1.3.21). Since $\underline{F}_N$ is inherently discontinuous across the folding lines according to the way it is constructed, as $N \to \infty$, $\underline{m}_N$ will oscillate about $\underline{j}$ at high frequencies.

Since the folding operation introduces no additional stress and no changes in the original stretches λ and $\nu(\lambda)$, the strain energy function $W(\underline{F}_N)$ must be equal to that of $w(\cdot)$ evaluated at the natural width. This is

$$W(\underline{F}_N) = w(\lambda,\tilde{\mu}) = w(\lambda,\nu(\lambda)) = w^R(\lambda), \quad \tilde{\mu} < \nu(\lambda), \quad N = 1,2,3,\ldots \tag{1.3.22}$$

where $w^R(\lambda)$ is the *Relaxed Strain Density* according to Pipkin [1.7]. It is important to point out that $w^R(\lambda)$ is not equal to that of $W(\cdot)$ evaluated at $\tilde{\underline{F}}$.

It can be concluded that any value of $\mu \in (0, \nu(\lambda)]$ may be achieved by the finely scaled continuous folding. As shown by (1.3.22), the strain energy associated with each member of the folding sequence is a constant and equal to $w^R(\lambda)$, thus, the relaxed strain

energy is independent of μ when $0 \leq \mu \leq \upsilon(\lambda)$. A similar construction can be used to obtain the smooth deformation with stretches $\mu > 1$ and $\lambda \in (0, \upsilon(\mu)]$ by finely scaled continuous folding parallel to μ direction. Finally, the state of stretches with $\lambda < 1$ and $\mu < 1$ can be achieved by limits of sequences of double folding along λ and μ directions simultaneously. If the strain energy associated with the unstrained unit square sheet ($\lambda = \mu = 1$) is zero, the relaxed strain energy for $\lambda < 1$ and $\mu < 1$ is identically zero as well.

Fig.1.3.2 shows the relations between the ordinary strain energy function $w(\lambda,\mu)$ and the corresponding relaxed strain energy $w^R(\lambda)$ as they are plotted against μ for fixed λ.

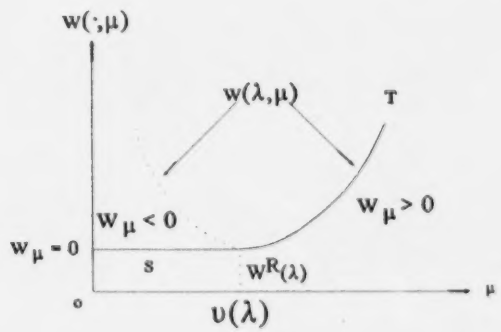

Fig.1.3.2 Relaxed strain energy function

In view of $(1.3.8)_4$, $w(\lambda,\mu)$ is convex with respect to μ for each fixed value of λ, and it approaches infinity as μ approaches zero or infinity. The minimum of $w(\lambda,\mu)$ with respect to μ is attained at the point $\mu = \upsilon(\lambda)$ which is assumed to be uniquely determined from the condition $w_\mu(\lambda, \cdot) = 0$. In the region $0 < \mu < \upsilon(\lambda)$, the dashed line represents

the value of original strain function $w(\lambda,\mu)$, whereas the horizontal line is the relaxed strain energy function $w^R(\lambda)$. In this region, $w^R(\lambda)$ is distinct from $w(\lambda,\mu)$ by the fact that $w^R_\mu(\lambda) = 0$, while $w_\mu(\lambda,\mu) < 0$. Similar discussions on the different behaviours of elastic material in tension and compression was given by Green [1.30].

The solid curve in Fig.1.3.2 collectively represents the behaviours of elastic membrane in distinct states of stress. When the membrane is in biaxial tension, $\mu > \upsilon(\lambda)$, its behaviour follows the T-curve. When the membrane is wrinkled, $\mu < \upsilon(\lambda)$, its behaviour is described by the S-curve. It is easy to show that $w^R(\lambda)$ satisfies $(1.3.8)_{2,4}$ and (1.3.9) identically.

Without resort to three dimensional elasticity, Pipkin [1.9] introduced the relaxed strain energy $w^R(\lambda)$ into the classical membrane theory which is defined in such a way that the principal stress resultants derived from it are never compressive for any admissible values of principal stretches. For the deformation with $0 \le \mu \le \upsilon(\lambda)$, instead of applying compressive stress $w_\mu < 0$, Pipkin [1.7] has suggested that smooth deformations be constructed as the limits of sequences of finely-wrinkled configurations involving closely-spaced folds parallel to λ-lines. Additional properties of the relaxed strain energy follow from the remaining inequalities $(1.3.8)_{1,3}$. In particular, if $\lambda > 1$ and $\mu \le \upsilon(\lambda)$, it is necessary that

$$w^R_\lambda(\lambda) \ge 0 \text{ and } w^R_{\lambda\lambda}(\lambda) \ge 0. \tag{1.3.23}$$

Thus all of the inequalities in (1.3.8, 9) are automatically satisfied for $\lambda > 1$, $\mu < \upsilon(\lambda)$. The latter restriction is consistent with experimental data on rubber [1.16].

Summarizing all properties associated with elastic membranes, the strain energy

function can be written in a composite form

$$\bar{w}(\lambda,\mu) = \begin{cases} w(\lambda,\mu), & \mu \geq \nu(\lambda),\ \lambda \geq \nu(\mu) \\ w^R(\lambda), & \mu \leq \nu(\lambda),\ \lambda \geq 1 \\ w^R(\mu), & \lambda \leq \nu(\lambda),\ \mu \geq 1 \\ 0, & \lambda < 1,\ \mu < 1 \end{cases} \tag{1.3.24}$$

Using the relaxed strain energy, we are able to reformulate the membrane theory so that it can account for finely scaled wrinkling automatically. This is accomplished simply by replacing the original strain energy function $W(\underline{F})$ in (1.2.24) by the composite energy function $\bar{w}(\lambda,\mu)$ (1.3.24). Constitutive relations obtained from $\bar{w}(\cdot)$ delivers positive principal stresses for $\lambda > \upsilon(\lambda)$ and $\mu > \upsilon(\lambda)$ in the region of (λ,μ)-plane. The membrane is *tense* in this region which is indicated by the first branch of (1.3.24). For stretches $\lambda > 1$, $0 \leq \mu \leq \upsilon(\lambda)$, the membrane may be regarded as finely wrinkled, in this region $\bar{w}(\cdot) = w^R(\lambda)$ determined by the second branch of (1.3.24). By symmetry, the relaxed energy is $w^R(\mu) = w(\upsilon(\mu),\mu)$ if $\mu > 1$, $0 \leq \lambda \leq \upsilon(\mu)$. The last of (1.3.24) corresponds to a state of zero stress. In this region, the membrane is wrinkled in both λ and μ directions.

In the subsequent analysis, we drop the overbar and use $w(\lambda,\mu)$ in the problem formulation. The use of relaxed strain energy is made explicit by writing $w^R(\lambda)$.

1.4. Tension field theory

When the relaxed energy $w^R(\lambda)$ is used in the membrane theory, deformations with stretches $\lambda > 1$ and $\mu \leq \upsilon(\lambda)$ furnish a state of stress that is locally uniaxial tension. Steigmann [1.12] has reformulated the *Tension Field Theory* so that the modified

membrane theory is able to account for wrinkling automatically. We only recount some important results from Steigmann's Tension Field Theory [1.12]. Detailed formulation of the theory can be found in [1.12,31,32,33].

Suppose that the membrane is wrinkled in the direction of μ, and substitute $(1.3.24)_2$ into (1.2.19) to obtain the Piola stress tensor

$$\underline{T} = w^R_\lambda(\lambda)\, \underline{l} \otimes \underline{L} \tag{1.4.1}$$

where the unit vector $\underline{L}$ and $\underline{l}$ are tangential to the tension lines (wrinkles) on the reference and deformed surfaces S and s, respectively. The stress vector is thus obtained as

$$\underline{T}^\alpha = w^R_\lambda(\lambda)\, L^\alpha\, \underline{l} \tag{1.4.2}$$

where $L^\alpha = \underline{L} \cdot \underline{A}^\alpha$. Here $\underline{A}^\alpha$ are the reciprocal base vector defined by (1.2.6).

The traction on the material element $d\underline{x}$ with unit normal $\underline{\nu}$ is calculated by equation (1.2.17) which takes the form

$$\underline{t} = \underline{T}\underline{\nu} = w^R_\lambda(\lambda)\, \underline{l}\, (\underline{L} \cdot \underline{\nu}). \tag{1.4.3}$$

This is an uniaxial tension in the direction of $\underline{l}$ on the deformed surface s. Therefore, the stresses in the wrinkled region form a tension field.

Substituting (1.4.2) into $(1.3.2)_1$, Steigmann [1.12] has derived the equations of equilibrium in the tension field

$$\begin{cases} A^{-\frac{1}{2}} \left[A^{\frac{1}{2}} w^R_\lambda L^\alpha \right]_{,\alpha} = 0, \\ k_g = 0, \quad k_n = -\dfrac{p^d \mu}{w^R_\lambda} \end{cases} \tag{1.4.4}$$

where k_g and k_n are the geodesic and normal curvatures, respectively, in the wrinkled

part of the deformed surface s. (see Eqns. 6.11, 6.12 of ref. [1.12]). The fact that $k_g = 0$ implies that tension lines are geodesics in the tension field. Equation $(1.4.4)_3$ relates the normal curvature k_n to the internal pressure p^d explicitly. If $p^d = 0$, $k_n = 0$ and, consequently, the tension field corresponds to a ruled surface. This result is obtained without particular reference to the form of relaxed strain energy function and the shape of the reference configuration.

According to Steigmann [1.12], $(1.4.4)_{2,3}$ can be combined to obtain an equation in terms of the deformation $\underline{r}$ (see Eqn. 6.15 of ref [1.12])

$$\lambda^{-1}(\underline{L}\cdot\underline{\nabla})[\lambda^{-1}(\underline{L}\cdot\underline{\nabla})\underline{r}] = -(\frac{p^d}{\lambda w^R{}_\lambda})(\underline{L}\cdot\underline{\nabla})\underline{r}\times(\underline{M}\cdot\underline{\nabla})\underline{r} \qquad (1.4.5)$$

while $(1.4.4)_1$ can be written in the tensor form (see Eqn. 6.13 of Ref [1.12])

$$\underline{\nabla}\cdot[w^R{}_\lambda(\lambda)\underline{L}] = 0, \qquad \underline{N}\cdot\underline{\nabla}\times(\lambda\underline{L}) = 0. \qquad (1.4.6)$$

To solve these equations, two solving functions $\{\phi,\psi\}$ were proposed in [1.12]. To show this, we assume that there exists a stretch potential function ϕ such that

$$\lambda\underline{L} = \underline{\nabla}\phi, \qquad (1.4.7)$$

where $\lambda = |\underline{\nabla}\phi|$. From (1.4.7), the gradient $\underline{\nabla}\phi$ is orthogonal to $\underline{M}$. It is also shown that a material element $d\underline{x}$ in the direction of $\underline{L}$ corresponds to the differential $d\phi = \lambda\underline{L}\cdot d\underline{x}$ which measures the arclength along tension lines on the deformed surface s.

Substituting $\lambda\underline{L}$ and $\underline{L}$ in terms of ϕ into equations (1.4.6), it is found that $(1.4.6)_2$ is satisfied identically, while $(1.4.6)_1$ furnishes an equation in ϕ

$$\underline{\nabla}\cdot[|\underline{\nabla}\phi|^{-1}w^R{}_\lambda(|\underline{\nabla}\phi|)\underline{\nabla}\phi] = 0 \qquad (1.4.8)$$

The traction on the boundary ∂S_t with unit normal $\underline{\nu}$ is given by (1.4.3):

$$\underline{F}^d = |\underline{\nabla}\phi|^{-1} w^R{}_\lambda(|\underline{\nabla}\phi|)(\underline{\nu}\cdot\underline{\nabla}\phi)\,\underline{l}. \tag{1.4.9}$$

This supplies the boundary condition for the solutions of ϕ obtained from (1.4.8). When the boundary condition is given in terms of the traction $\underline{F}^d$, the value of λ in terms of $\underline{F}^d$ on ∂S_t obtained from (1.4.9) is not uniquely determined. To resolve the non-uniqueness issue, it is necessary that some auxiliary conditions must be supplemented to determine the solution. For example, minimize the potential energy with respect to the competing values.

To complete the set of solving functions, a force potential function ψ is introduced such that $(1.4.6)_1$ becomes an identity. The force potential is defined by the relation

$$\underline{\nabla}\psi = w^R{}_\lambda(\lambda)\underline{M} \quad \text{or} \quad w^R{}_\lambda(\lambda)\underline{L} = \underline{\nabla}\psi \times \underline{N}; \tag{1.4.10}$$

where $w^R{}_\lambda(\lambda) = |\underline{\nabla}\psi|$. $(1.4.10)_1$ implies that $\underline{\nabla}\psi$ is orthogonal to $\underline{L}$. It is a straightforward matter to verify that equation $(1.4.6)_1$ is an identity if $w^R{}_\lambda(\lambda)\underline{L}$ is replaced by $(1.4.10)_2$. $(1.4.6)_2$ furnishes an equation in ψ

$$\underline{\nabla}\cdot(h(|\underline{\nabla}\psi|)\underline{\nabla}\psi) = 0 \tag{1.4.11}$$

where $h(|\underline{\nabla}\psi|) = \lambda(|\underline{\nabla}\psi|) / |\underline{\nabla}\psi|$ and $\lambda(|\underline{\nabla}\psi|)$ is the root of equation $w^R{}_\lambda(\lambda) = |\underline{\nabla}\psi|$.

It may be seen that a material element $d\underline{x}$ in the direction of $\underline{M}$ corresponds to the differential $d\psi = w^R{}_\lambda(\lambda)\underline{M}\cdot d\underline{x}$ which is the total force channelled between neighbouring tension lines on the deformed surface s. From the defining equation $(1.4.10)_1$, $w^R{}_\lambda(\lambda) = \underline{\nabla}\psi\cdot\underline{M}$. Using this result in (1.4.3), the boundary condition is

$$\underline{F}^d = (\underline{\nabla}\psi\cdot\underline{s})\,\underline{l}. \tag{1.4.12}$$

where $\underline{s}$ is the tangent to the boundary curve on ∂S_t on which $\underline{F}^d$ is applied.

From the defining equations (1.4.7) and $(1.4.10)_1$, the functions $\phi(\theta^\alpha)$ and $\psi(\theta^\alpha)$ form

an orthogonal curvilinear coordinate system on the reference surface S. ψ-curve coincides with the tension lines on which different points are distinguished by their values of arclength ϕ. For the coordinate transformation $(\theta^1,\theta^2) \rightarrow (\phi,\psi)$, the Jacobian matrix is

$$\underline{N} \cdot \underline{\nabla}\phi \times \underline{\nabla}\psi = \lambda w^R{}_\lambda(\lambda) \tag{1.4.13}$$

which is positive since $w^R{}_\lambda(\lambda) > 0$ determined from $(1.3.23)_1$ in the tension field. This ensures the functional independence of ϕ and ψ. Taking $\{\phi,\psi\}$ as independent variables, the gradient operator is defined in the general form

$$\underline{\nabla}(\ ) = \partial(\ )/\partial\phi\ \underline{\nabla}\phi + \partial(\ )/\partial\psi\ \underline{\nabla}\psi \tag{1.4.14}$$

Substituting (1.4.7), $(1.4.10)_1$ into the above equation, we have

$$\underline{\nabla}(\ ) = \partial(\ )/\partial\phi\ \lambda\underline{L} + \partial(\ )/\partial\psi\ w^R{}_\lambda(\lambda)\underline{M}. \tag{1.4.15}$$

Scalar-multiplying the above equation by $\underline{L}$ and $\underline{M}$ respectively, the directional derivatives in terms of ϕ and ψ are

$$\lambda^{-1}(\underline{L} \cdot \underline{\nabla}) = \partial\ /\ \partial\phi,\ \ w^R{}_\lambda(\lambda)^{-1}(\underline{M} \cdot \underline{\nabla}) = \partial\ /\ \partial\psi \tag{1.4.16}$$

Upon substitution of (1.4.16) into equation (1.4.5), the equilibrium equation of deformation in the tension field is simply

$$\underline{r}_{\phi\phi} = -\ p^d\ \underline{r}_\phi \times \underline{r}_\psi. \tag{1.4.17}$$

In particular, if $p^d = 0$, the right-hand side vanishes identically, and we have a homogeneous differential equation of second order:

$$\underline{r}_{\phi\phi} = 0. \tag{1.4.18}$$

Integration of (1.4.18) yields the equation

$$\underline{r}_\phi = \underline{a}(\psi). \tag{1.4.19}$$

where $\underline{a}$ is an arbitrary function of ψ. From $(1.4.16)_1$, we have

$$\underline{r}_\phi = \lambda^{-1} (\underline{I} \cdot \underline{\nabla})\underline{r} = \underline{l}. \tag{1.4.20}$$

Here $\underline{r}_\phi$ is the derivative of $\underline{r}$ with respect to the arclength ϕ along the tension line. Comparing (1.4.20) with (1.4.19), it concludes that $\underline{a} = \underline{l}(\psi)$ and (1.4.20) becomes

$$\underline{r}_\phi = \underline{l}(\psi) \tag{1.4.21}$$

Integration of (1.4.21) gives the deformation in the tension field

$$\underline{r} = \phi \underline{l}(\psi) + \underline{u}(\psi) \tag{1.4.22}$$

where $\underline{u}(\psi)$ is an arbitrary vector function to be determined by prescribed displacement on ∂S_b. It is obvious that (1.4.22) is the equation of a ruled surface generated by a one-parameter family of unit vectors $\underline{l}(\psi)$.

Determinations of ϕ and ψ involves solving differential equations (1.4.8) and (1.4.11), and the solution must satisfy the boundary conditions (1.4.9) and (1.4.12).

Deformation (1.4.22) will be used in Chapter 2 to construct the solution in the wrinkled region of an annular membrane subjected to an axisymmetric twist.

CHAPTER 2 FINITE PALNE TWIST OF AN ANNULAR MEMBRANE

Partial wrinkling analysis of plane axisymmetric twist of an annular membrane was previously studied by Reissner [1.2] and Stein and Hedgepeth [1.4] using the infinitesimal deformation theory. In this section, we extend the solutions to the finite deformations. The problem to be considered is a lateral traction free annular membrane whose outer boundary is fixed and prestretched by attaching its inner boundary to a rigid hub. The hub is then rotated through a prescribed angle. The deformed membrane is partly wrinkled, which occurs in an annular band adjacent to the hub. General solutions of the deformation are obtained using tension field theory. In particular, in the absence of lateral traction, the general solution in the wrinkled region is available in equation (1.4.23). Kinematic relations associated with the axisymmtric deformations of an annular membrane are derived in section 2.1. Several geometrical variables are defined to facilitate further formulations. In section 2.2, the strain energy function is specified as the particular type of harmonic material. Relaxed strain energy density based on it is derived. In section 2.3, we obtain the deformation in the tense region and use these results to analyze the influence of geometry of the reference annulus and dimension of the hub. A set of conditions are thus obtained which indicate the state of incipient wrinkling and the limiting state. In section 2.4, deformation (1.4.23) for the wrinkled region is specialized to conform to the particular boundary conditions. General solutions are constructed, in section 2.5, by matching the solutions obtained in section 2.3 for tense region and the solutions obtained in section 2.4 for partly wrinkled regions, at the common boundary. Non-dimensionalization of the solving equations is performed in

section 2.6 and the numerical solutions are presented in section 2.7.

2.1 Kinematics of plane axisymmetric deformation with twist

In this section, we consider axisymmetric deformations of an annular membrane in its own plane. The membrane is fixed at its outer boundary while its inner boundary is attached to a concentric rigid hub. The lateral surface of the membrane is assumed to be traction free as shown in Fig. 2.1.1. The deformation under consideration is created by the rotation of the hub through a certain angle about the axis of symmetry. For the sake of convenience, we choose a two-dimensional polar coordinate system (r,θ) with respect to which geometries of the reference and deformed surfaces are analyzed. In this setting, an undeformed annular surface with outer radius r_o and inner radius r_i occupies a region of an annulus Ω: $r_i \le r \le r_0$, $0 \le \theta \le 2\pi$ with the unit normal vector $\underline{k}$. In the sequel, we use Ω to denote the reference surface, and coordinates (r,θ) to label the material particle on Ω. Thus, (r,θ) are regarded as independent variables in the formulation. The position of a material particle on Ω is given by the vector,

$$\underline{x} = r\,\underline{i}(\theta), \tag{2.1.1}$$

where $\underline{i}(\theta)$ is a unit vector in the radial direction, θ is the azimuthal angle. The reference surface Ω is illustrated in Fig. 2.1.2.

The directional difference in the position of two neighbouring particles is described by the differential vector $d\underline{x}$ which, by differentiating (2.1.1) with respect to (r,θ), is given by

$$d\underline{x} = \underline{i}(\theta)\,dr + r\,\underline{j}(\theta)\,d\theta \tag{2.1.2}$$

where $\underline{j}(\theta) = \underline{i}'(\theta)$ is orthogonal to $\underline{i}(\theta)$. The corresponding increments in the independent coordinates (r,θ) are obtained by scalar-multiplying $d\underline{x}$ in (2.1.2) with $\underline{i}(\theta)$, $\underline{j}(\theta)$, respectively, they are obtained in the form

$$dr = \underline{i}(\theta) \cdot d\underline{x}; \quad d\theta = r^{-1}\, \underline{j}(\theta) \cdot d\underline{x} \tag{2.1.3}$$

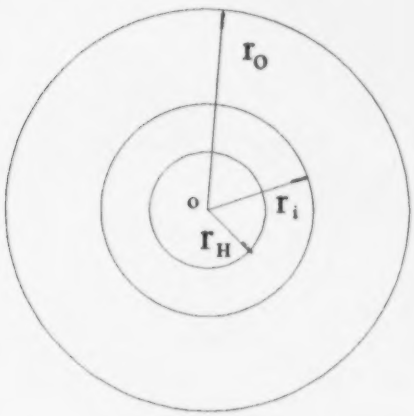

Fig.2.1.1 Reference annulus and the Hub

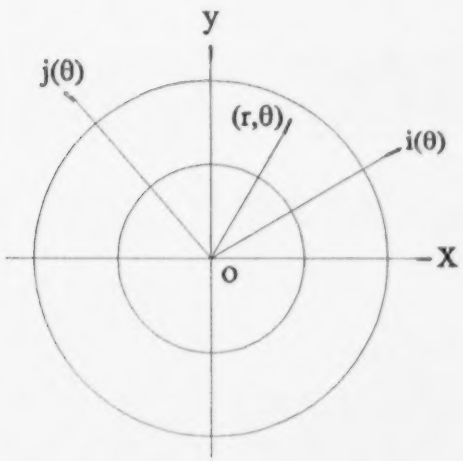

Fig.2.1.2 Polar coordinate system

A deformation of Ω is defined as a smooth, one-to-one mapping $\underline{y}(\underline{x})$ which maps Ω onto a new configuration. In a planar deformation in which the membrane is axisymmetrically twisted, the particle with coordinates (r,θ) in Ω is displaced to the new place on the deformed surface which is described by the position vector

$$\underline{y}(\underline{x}) = \rho(r)\underline{i}(\theta + \Gamma(r)), \tag{2.1.4}$$

where $\rho(r)$ is the deformed radius and Γ is the twist angle measured counterclockwisely from its reference radius $\underline{i}(\theta)$. In axisymmetric deformations, ρ and Γ are independent of θ. Thus, determination of functions $\rho(r)$ and $\Gamma(r)$ are the primary task of the work included in this chapter.

It is convenient to write the deformation $\underline{y}$ in the form

$$\underline{y}(\underline{x}) = u(r)\underline{i}(\theta) + v(r)\underline{j}(\theta) \tag{2.1.5}$$

with components $u(r) = \rho(r) \cos \Gamma(r)$, $v(r) = \rho(r) \sin \Gamma(r)$.

Local deformation is described by the deformation gradient $\underline{F}(\underline{x})$. An expression for $\underline{F}(\underline{x})$ may be obtained in a way described as follows. We first differentiate (2.1.5) to write $d\underline{y}(\underline{x})$ as a linear combination of dr and $d\theta$:

$$d\underline{y} = (u'(r)\underline{i} + v'(r)\underline{j})\, dr + (u(r)\underline{j} - v(r)\underline{i})\, d\theta. \tag{2.1.6}$$

where the prime denotes the differentiation with respect to r. $\underline{j}'(\theta) = -\underline{i}'(\theta)$ has been used. Secondly, we substitute dr, $d\theta$ in (2.1.3) into (2.1.6) and rearrange the relevant terms to write the result in the form of the defining equation (1.2.15) of $\underline{F}$, then (2.1.6) shows that

$$\underline{F} = u'(r)\underline{i}\otimes\underline{i} + v'(r)\underline{j}\otimes\underline{i} + (u/r)\underline{j}\otimes\underline{j} - (v/r)\underline{i}\otimes\underline{j}\,, \tag{2.1.7}$$

For deformations confined to a plane, components of $\underline{F}$ is a 2×2 square matrix and

thus $\underline{F}$ is uniquely invertible provided $\det(\underline{F}) > 0$. From (2.1.7), its transpose $\underline{F}^T$ and inverse $\underline{F}^{-1}$ may be written in the form

$$\underline{F}^T = u'(r)\underline{i}\otimes\underline{i} + v'(r)\underline{i}\otimes\underline{j} + (u/r)\underline{j}\otimes\underline{j} - (v/r)\underline{j}\otimes\underline{i} \tag{2.1.8}$$

$$\underline{F}^{-1} = [\det(\underline{F})]^{-1}[(u/r)\underline{i}\otimes\underline{i} - v'(r)\underline{j}\otimes\underline{i} + u'(r)\underline{j}\otimes\underline{j} + (v/r)\underline{i}\otimes\underline{j}] \tag{2.1.9}$$

where $\det(\underline{F})$ is the determinant of $\underline{F}$. Equations (2.1.8, 9) will be needed later.

The change in the modulus of differential vector $d\underline{x}$ is measured by the Cauchy-Green strain tensor $\underline{C} = \underline{F}^T\underline{F}$ which, with the substitutions of $\underline{F}$ in (2.1.7) and $\underline{F}^T$ in (2.1.8), takes on the form

$$\underline{C} = [(u')^2 + (v')^2]\,\underline{i}\otimes\underline{i} + [(u/r)^2 + (v/r)^2]\,\underline{j}\otimes\underline{j} + [(u/r)v' - (v/r)u']\,(\underline{i}\otimes\underline{j} + \underline{j}\otimes\underline{i}). \tag{2.1.10}$$

As mentioned previously, we are assuming that the membrane is made of homogeneous isotropic materials endowed with a strain energy function $w(\cdot)$. The membrane possessing such properties is characterized by the strain energy function that depends on strain tensor $\underline{C}$ through the determinant $\det(\underline{C})$ and the trace $\mathrm{tr}(\underline{C})$ which are, in view of relations (1.2.25), in one-to-one correspondence with the invariants I and J. Since $\underline{F}$ is invertible in a planar deformation, then it follows from the definition $(1.2.25)_1$ and (2.1.10) that

$$J = \det(\underline{F}) = [\det(\underline{C})]^{1/2} = (u(r)/r)\,u'(r) + (v(r)/r)\,v'(r). \tag{2.1.11}_a$$

Use of (2.1.5) in $(2.1.11)_a$ results in an alternative form

$$J = (1/2)\,r^{-1}\,(\underline{y}\cdot\underline{y})' \tag{2.1.11}_b$$

By definition of $(1.2.25)_2$, together with relations (2.1.10) and $(2.1.11)_a$, the invariant I is obtained as the square root:

$$I = [(u'(r) + u(r)/r)^2 + (v'(r) + v(r)/r)^2]^{1/2}. \qquad (2.1.12)$$

which is positive by definition.

In membrane theory, Pipkin [1.7] has proved that deformation gradient $\underline{F}$ can be written in the form of (1.2.22) in which λ, μ are principal stretches and $\{\underline{L},\underline{M}\}$ and $\{\underline{l},\underline{m}\}$ are the associated principal vectors on the reference and deformed surfaces. For the problem in which the initial and deformed surfaces remain in the same plane, principal vectors $\{\underline{L},\underline{M}\}$ and $\{\underline{l},\underline{m}\}$ can be resolved with respect to the same orthonormal base $\{\underline{i}(\theta), \underline{j}(\theta)\}$. To this end, let γ, α be the angles formed between $\underline{i}(\theta)$ and the principal vectors $\underline{L}$, $\underline{l}$ appearing in equation (1.2.22), we have

$$\left.\begin{aligned} \underline{L} &= \cos\gamma\, \underline{i}(\theta) + \sin\gamma\, \underline{j}(\theta), \quad \underline{l} = \cos\alpha\, \underline{i}(\theta) + \sin\alpha\, \underline{j}(\theta), \\ \underline{M} &= -\sin\gamma\, \underline{i}(\theta) + \cos\gamma\, \underline{j}(\theta), \quad \underline{m} = -\sin\alpha\, \underline{i}(\theta) + \cos\alpha\, \underline{j}(\theta). \end{aligned}\right\} \qquad (2.1.13)$$

Fig. 2.1.3 shows the orientations of angles γ and α.

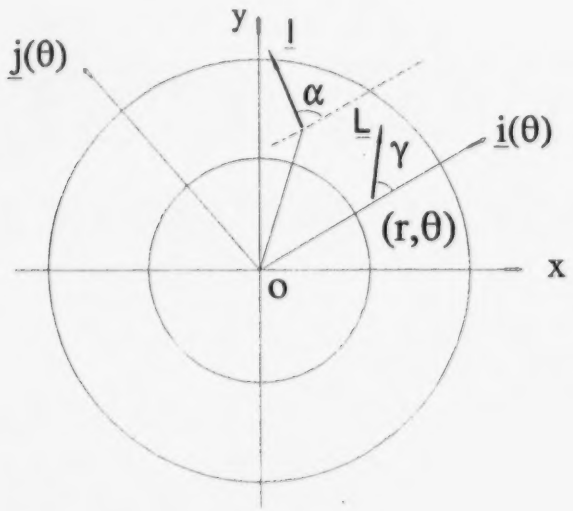

Fig. 2.1.3 Orientations of principal vectors

Substituting (2.1.13) into (1.2.22), $\underline{F}$ can be written in the form

$$\left.\begin{aligned}\underline{F} = {} & (\lambda \cos\alpha \cos\gamma + \mu \sin\alpha \sin\gamma)\, \underline{i}\otimes\underline{i} + (\lambda \cos\alpha \sin\gamma - \mu \sin\alpha \cos\gamma)\, \underline{i}\otimes\underline{j} \\ & + (\lambda \sin\alpha \sin\gamma + \mu \cos\alpha \cos\gamma)\, \underline{j}\otimes\underline{j} + (\lambda \sin\alpha \cos\gamma - \mu \cos\alpha \sin\gamma)\, \underline{j}\otimes\underline{i}.\end{aligned}\right\} \tag{2.1.14}$$

A straightforward comparison of $\underline{F}$ in (2.1.14) with that expressed in (2.1.7) enables us to obtain the relations

$$\left.\begin{aligned}(\lambda + \mu)\exp(i\omega) &= u' + u/r + i(v' + v/r), \\ (\lambda - \mu)\exp(i\Omega) &= u' - u/r + i(v' - v/r);.\end{aligned}\right\} \tag{2.1.15}$$

where $\omega = \alpha - \gamma$, $\Omega = \alpha + \gamma$ and $i=\sqrt{-1}$. exp( ) denotes the exponential function.

Relations (2.1.15) can be used to calculate λ, μ, α and γ in terms of deformation components u, v, u´ and u´. A simple mathematical manipulation based on (2.1.15) then yields expressions for λ and μ. They are

$$\left.\begin{aligned}2\lambda &= ((u'+u/r)^2+(v'+v/r)^2)^{1/2} + ((u'-u/r)^2+(v'-v/r)^2)^{1/2} \\ 2\mu &= ((u'+u/r)^2+(v'+v/r)^2)^{1/2} - ((u'-u/r)^2+(v'-v/r)^2)^{1/2}\end{aligned}\right\} \tag{2.1.16}$$

and, in addition, also an expression for angle ω

$$\omega = \alpha - \gamma = \cos^{-1}\left[\frac{u'+u/r}{\left[(u'+u/r)^2+(v'+v/r)^2\right]^{1/2}}\right] \tag{2.1.17}$$

which determines the rotation of material element aligned with the principal directions: $\underline{l} = \cos\omega\, \underline{L} + \sin\omega\, \underline{M}$. Moreover, we are able to obtain the equation for the angle α:

$$2\alpha = \cos^{-1}\left[\frac{u'+u/r}{\left[(u'+u/r)^2+(v'+v/r)^2\right]^{\frac{1}{2}}}\right] - \cos^{-1}\left[\frac{u'-u/r}{\left[(u'-u/r)^2+(v'-v/r)^2\right]^{\frac{1}{2}}}\right] \tag{2.1.18}$$

2.2 Harmonic material

We suppose that the membrane is composed of the *harmonic* materials which can be described by the function

$$w(\lambda,\mu) = W(I,J) = 2G[F(I) - J] , \qquad (2.2.1)$$

where G is a positive constant with dimensions of force/length and $F(\cdot)$ is a C^2 function. I, J are invariants of the Cauchy-Green strain tensor $\underline{C}$ defined by (1.2.26). Harmonic material (2.2.1) has been used previously by John [2.1] in the context of plane strains. As shown in section 2.3, use of harmonic material can simplify the equations of equilibrium considerably so that it is possible, in our case, that analytical solutions can be obtained without resort to numerical computations. In fact, as shown in section 2.4, for the harmonical material specialized in (2.2.3), the deformation for the tense part of the membrane can be obtained analytically.

The function F(I) is expected to satisfy certain inequalities in order that the solutions obtained conform with the physical realities. With the help of relations (1.2.26), it can be easily verified that the function $w(\lambda,\mu)$ defined by equation (2.2.1) satisfies inequalities $(1.3.10)_{3,4,5}$ and (1.3.11) if and only if function F(I) satisfies the following conditions

$$F'(I) \geq 0 \quad \text{and} \quad F''(I) \geq 0. \qquad (2.2.2)$$

In particular, we consider a special type of harmonic material such that the uniaxial force-stretch relation $f(\lambda)$ and the natural width $\upsilon(\lambda)$ (1.3.17) are given by

$$f(\lambda) = 2G(\lambda - \lambda^{-1}), \qquad \upsilon(\lambda) = \lambda^{-1} \qquad (2.2.3)$$

Harmonic material defined by (2.2.3) has been used by Varley and Cumberbatch [2.2]

in studying the problems of thin sheets with elliptical holes. Solutions were shown to have good agreement with experimental data on rubber [2.2].

It remains to derive the functional form of F(I) for the given uniaxial constitutive relation $(2.2.3)_1$. We recall that $f(\lambda) = w_\lambda$ is the stress resultant in the state of uniaxial tension. In this state, the transverse stress resultant $w_\mu = 0$. Using the chain rule of differentiation, with the aid of relations (1.2.26), the principal stresses are

$$w_\lambda = W_I I_\lambda + W_J J_\lambda = 2G[F'(I) - \mu],$$

$$w_\mu = W_I I_\mu + W_J J_\mu = 2G[F'(I) - \lambda]. \qquad (2.2.4)$$

By setting w_μ in $(2.2.4)_2$ to zero at the natural width, i.e. $I = \lambda + v(\lambda) = \lambda + \lambda^{-1}$, we have

$$F'(I) = I/2 + [(I/2)^2 - 1]^{1/2}, \qquad (2.2.5)$$

where the right-hand side of (2.2.5) is λ which is expressed in terms of I as a root of equation $\lambda^2 - \lambda I + 1 = 0$. The equation has real solutions if and only if $I/2 \geq 1$, and only one of them makes the function $F'(\cdot)$ to satisfy $(2.2.2)_2$ when $I/2 > 1$.

For the given natural width $(2.2.3)_2$, in view of $(1.2.26)_2$, $J > 1$ implies that $\lambda > v(\mu)$ and $\mu > v(\lambda)$. According to equation (1.2.26), $I^2 - 4J = (\lambda - \mu)^2 > 0$, so that $I/2 > J > 1$ holds in this region. Then (2.2.5) holds true and the strict inequalities in (2.2.2) are satisfied. The second of (2.2.2) implies that $w_{\lambda\lambda} > 0$, $w_{\mu\mu} > 0$, since $w_\mu = 0$ at $\mu = v(\mu)$, it follows that $w_\mu > 0$ for $\mu > v(\lambda)$. Since $w(\lambda,\mu)$ is symmetric with respect to λ and μ, we have $w_\lambda > 0$ for $\lambda > v(\mu)$. Thus all of the strict inequalities in (1.3.10), (1.3.11) are satisfied for $\lambda > v(\mu)$ and $\mu > v(\lambda)$. In the wrinkled region $\lambda > 1$, $0 \leq \mu \leq v(\lambda)$, $w_\lambda = f(\lambda) > 0$, $w_\mu = 0$, whereas $w_\lambda = 0$, $w_\mu = f(\mu) > 0$ in the region $\mu >$

1, $0 \leq \lambda \leq \upsilon(\mu)$. In the former case, $(1.3.10)_2$ is satisfied as an identity. In latter case, $(1.3.10)_1$ is satisfied identically. With the energy identically zero for $0 \leq (\lambda,\mu) \leq 1$, it can now be verified that the Legendre-Hadamard inequality is satisfied for all $\lambda, \mu \geq 0$.

The relaxed strain energy $\hat{W}(\cdot)$ is also locally convex in the sense that the second derivative of $\hat{W}(\underline{F} + \varepsilon\underline{A})$, evaluated at $\varepsilon = 0$, is non-negative for every $\underline{A}$. To see this we use Pipkin's necessary and sufficient conditions for local convexity [1.7]. In addition to inequalities (1.3.10), the following inequality must be satisfied:

$$w_{\lambda\lambda}\, w_{\mu\mu} - w_{\lambda\mu}{}^2 \geq 0, \quad |a| \geq b, \tag{2.2.6}$$

where a and b are defined in (1.3.10). For the harmonic strain energy $(2.2.3)_2$, $(2.2.6)_{1,2}$ are equivalent to

$$F''(I) \geq 1/2, \qquad F'(I) \geq I/2 \tag{2.2.7}$$

respectively. Thus the relaxed strain energy defined by (2.2.3) which satisfies (2.2.5) is convex for all $\lambda > 0$, $\mu \geq 0$ and strictly convex if $\lambda > \upsilon(\mu)$ and $\mu > \upsilon(\lambda)$.

For harmonic material with stress-strain relations like (2.2.4), the Piola stress tensor $\underline{T}(\underline{F})$ defined in (1.2.37) takes on the form

$$(2G)^{-1}\underline{T} = dF/dI\ (\underline{l} \otimes \underline{L} + \underline{m} \otimes \underline{M}) - (\mu\, \underline{l} \otimes \underline{L} + \lambda\, \underline{m} \otimes \underline{M}). \tag{2.2.8}$$

We make an effort to eliminate λ and μ in (2.2.8). To this end, we use equation (1.2.22) to derive the inverse $\underline{F}^{-1}$ of $\underline{F}$ in terms of the principal stretches and principal vectors, it is

$$\underline{F}^{-1} = \lambda^{-1}\, \underline{L} \otimes \underline{l} + \mu^{-1}\, \underline{M} \otimes \underline{m}. \tag{2.2.9}$$

The transpose $\underline{F}^{-T}$ of $\underline{F}^{-1}$ is simply

$$\underline{F}^{-T} = \lambda^{-1}\, \underline{l} \otimes \underline{L} + \mu^{-1}\, \underline{m} \otimes \underline{M}. \qquad (2.2.10)$$

Multiplying (2.2.10) by J in $(1.2.26)_2$, we obtain the relation

$$J\underline{F}^{-T} = \mu\, \underline{l} \otimes \underline{L} + \lambda\, \underline{m} \otimes \underline{M} \qquad (2.2.11)$$

which is used to derive the equation

$$I^{-1}\,(\underline{F} + J\underline{F}^{-T}) = (\underline{l} \otimes \underline{L} + \underline{m} \otimes \underline{M}). \qquad (2.2.12)$$

in which $(1.2.26)_1$ for I has been used. By substituting (2.2.11) and (2.2.12) into (2.2.8), the Piola stress tensor becomes

$$(2G)^{-1}\underline{T} = dF/dI\; I^{-1}\,(\underline{F} + J\underline{F}^{-1}) - J\underline{F}^{-1}. \qquad (2.2.13)$$

Thus $\underline{T}$ is written in terms of $\underline{F}$ and the invariants I, J. Equation (2.2.13) is obtained without reference to the deformation, nor to the surface geometry, therefore, it is valid for any plane deformation of membrane made of harmonic materials of the special form of (2.2.3).

2.3 Tense solution and incipient wrinkling

The general solutions of axisymmetric equilibrium deformations with twist in fully tense parts of the deformed membrane can be obtained by using equilibrium equations (1.3.4) combined with equation (2.1.15) to derive a system of differential equations for the deformation components u(r) and v(r), in which w_λ and w_μ are given by (2.2.4). As an alternative, if the membrane is under the action of conservative forces, using the minimum energy principle, equations of equilibrium can also be obtained from the total potential energy of the deformation. Since the strain energy function depends on the invariants I and J, i.e $W(\cdot) = W(I,J)$, in view of (2.1.11, 12), $W(\cdot)$ is a certain

function of u, v, u´ and v´, i.e. W(I,J) = w(u, v, u´, v´). The membrane surface is free of traction and no prescribed edge forces along the inner and outer boundaries. The potential energy E[y] associated with the deformation can be written as an integral

$$E[\underline{y}] = 2\pi \int_{r_i}^{r_o} H(u,v;u',v')\, r\, dr, \tag{2.3.1}$$

where

$$H = w(I(u,v;u',v'), J(u,v;u',v')). \tag{2.3.2}$$

The torque which is needed on the hub to maintain the equilibrium will be calculated after the deformation is obtained. Thus, the problem posed is a pure displacement boundary-value problem.

For the potential functional (2.3.1), by following the familiar variational theory, we obtain the Euler-Lagrangean equations in the form

$$r\, H_u = (r\, H_{u'})'; \quad r\, H_v = (r\, H_{v'})', \tag{2.3.3}$$

where subscripts denote partial derivatives with respect to the indicated variables and the prime denotes the differentiation with respect to r.

To facilitate the further formulation, we establish the following relations. From equations $(2.1.11)_a$ and (2.1.12), we have partial derivatives

$$\left.\begin{aligned} J_u &= u'/r, \quad J_{u'} = u/r, \\ I_u &= (rI)^{-1}(u' + u/r), \quad I_{u'} = I^{-1}(u' + u/r), \end{aligned}\right\} \tag{2.3.4}$$

$$\left.\begin{aligned} J_v &= v'/r, \quad J_{v'} = v/r, \\ I_v &= (rI)^{-1}(v' + v/r), \quad I_{v'} = I^{-1}(v' + v/r), \end{aligned}\right\} \tag{2.3.5}$$

These can be used in expressions

$$H_u = w_1 I_u + W_J J_u, \quad H_{u\cdot} = w_1 I_{u\cdot} + w_J J_{u\cdot}, \tag{2.3.6}$$

to derive equations

$$\left.\begin{aligned}(rH_{u\cdot})' &= rH_u + r[I^{-1} W_1(u' + u/r)]' + (W_J)'u,\\ (rH_{v\cdot})' &= rH_v + r[I^{-1} W_1(v' + v/r)]' + (W_J)'v.\end{aligned}\right\} \tag{2.3.7}$$

Upon substitution of the right-hand side of equations $(2.3.7)_{1,2}$ into the left side of $(2.3.3)_{1,2}$ for $(rH_{u\cdot})'$ and $(rH_{v\cdot})'$, we arrive at a set of equilibrium equations of the form

$$\left.\begin{aligned}&[I^{-1} w_1(u' + u/r)]' + (w_J)' u/r = 0;\\ &[I^{-1} w_1(v' + v/r)]' + (w_J)' v/r = 0.\end{aligned}\right\} \tag{2.3.8}$$

Note that these equations are derived without restrictions to the forms of strain energy functions and thus they are valid for arbitrary isotropic materials. The governing equations of displacement (2.3.8) are completed once the functional form of the strain energy function $W(\cdot)$ appropriate to the material of interest is specified.

For the problem under consideration, we are particularly interested in the harmonic material specified by (2.2.1) which shows that $W(\cdot)$ is linear in J, thus, $W_J = -2G$ is a constant. This implies that $(W_J)' = 0$. As a result, the second terms on the left side of equations $(2.3.8)_{1,2}$ vanish identically and the first terms can be integrated once to yield two equations

$$\left.\begin{aligned}&I^{-1} (u' + u/r) F'(I) = C_1,\\ &I^{-1} (v' + v/r) F'(I) = C_2.\end{aligned}\right\} \tag{2.3.9}$$

C_1 and C_2 are integration constants. On squaring and adding $(2.3.9)_{1,2}$, and using equation (2.1.12), we obtain the equation

$$F'(I) = (C_1^2 + C_2^2)^{1/2}. \tag{2.3.10}$$

We choose positive root in accordance with restriction $(2.2.2)_1$ on $F'(I)$ in equilibrium.

Equation (2.3.10) requires that $F'(I)$ be a positive constant in equilibrium. Since we are considering material for which $F''(I) > 0$ as shown in (2.2.2), the condition for I to satisfy both conditions simultaneously is

$$I = \text{const.} \tag{2.3.11}$$

in the equilibrium equations $(2.3.9)_{1,2}$.

Since both I and $F'(I)$ are constants, (2.3.9) becomes a set of two independent ordinary differential equations of the first order in u and v. They can be integrated separately by the simple technique. This leads to the solutions of deformation components u and v:

$$u(r) = a_1 r + b_1/r, \quad v(r) = a_2 r + b_2/r, \tag{2.3.12}$$

in which integration constants a_i, b_i $(i=1,2)$ are to be determined once the appropriate boundary conditions for the given problem are prescribed. Solution (2.3.12) is valid for the annular region Ω: $r_i \le r \le r_0$, $0 \le \theta \le 2\pi$ as long as $r \neq 0$. This is secured by the presence of a central rigid hub which has a definite radius. Ogden and Isherwood [2.4] used a complex-variable formulation to demonstrate that a wide variety of boundary-value problems for harmonic materials can be solved by taking the invariant I to be constant.

Principal stretches λ and μ can be calculated by substituting the general deformation (2.3.12) into equations $(2.1.16)_{1,2}$. Taking $\lambda \ge \mu$ for definiteness, we obtain

$$\left.\begin{aligned} \lambda &= (a_1^2 + a_2^2)^{1/2} + r^{-2}(b_1^2 + b_2^2)^{1/2}, \\ \mu &= (a_1^2 + a_2^2)^{1/2} - r^{-2}(b_1^2 + b_2^2)^{1/2}, \end{aligned}\right\} \tag{2.3.13}$$

Moreover, the general forms of the invariants I, J are obtained by substituting (2.3.12)

into (2.1.11), and (2.1.12), we have

$$\left.\begin{aligned} J &= a_1^2 + a_2^2 - r^{-4}(b_1^2 + b_2^2), \\ I &= 2\,(a_1^2 + a_2^2)^{1/2}, \end{aligned}\right\} \qquad (2.3.14)$$

$(2.3.14)_2$ shows that I is a constant. Alternatively, relations (2.3.14) can be obtained using (1.3.25) together with (2.3.12) and (2.1.10).

For harmonic materials with the natural width $\nu(\lambda)$ defined by $(2.2.3)_2$, the membrane is tense if and only if $J > 1$. From equation $(2.3.14)_1$, it may be seen that J is a monotone increasing function of r in the interval $[r_i, r_0]$. Hence, deformation (2.3.12) is valid in the annulus $(\bar{r}, r_o]$, where

$$\bar{r} = [(b_1^2 + b_2^2)/(a_1^2 + a_2^2)]^{1/4} \qquad (2.3.15)$$

is the value of r at which $J = 1$. It is obtained by setting $J = 1$ in $(2.3.14)_1$. The circle $r = \bar{r}$ is an interior boundary separating the tense and wrinkled parts of the deformed membrane.

We now turn our attention to the problem posed in the beginning of this chapter in which an annular membrane is axisymmetric deformed with its outer boundary $r = r_o$ fixed and the inner boundary $r = r_i$ is attached to a rigid hub of radius ρ_i which assumes values less than or equal to r_i, i.e. $\rho_i \leq r_i$. This setting is shown in Fig. 2.1.2. Rotation of the hub through a prescribed angle τ about the axis of symmetry $\underline{k}$ produces the axisymmetric deformations of the membrane. In this case, boundary conditions of displacement on the outer and inner boundaries are given by

$$\begin{aligned} u_i &= \rho_i \cos\tau, \quad v_i = \rho_i \sin\tau, \quad \text{at } r = r_i \\ u_o &= r_o, \quad v_o = 0, \quad \text{at } r = r_o \end{aligned} \qquad (2.3.16)$$

We use the notation $u_o = u(r_o)$, $u_i = u(r_i)$, $v_o = v(r_o)$ and $v_i = v(r_i)$.

We first consider such ρ_i and τ that the entire membrane is tense, deformation corresponding to the tense region $r_i \le r \le r_o$ is determined completely by solution (2.3.12) in which constants a_1, a_2, b_1 and b_2 are determined through the prescription of boundary conditions (2.3.16) in (2.3.12)

$$\left.\begin{aligned} a_1 &= (r_o^2 - u_i\, r_i)/(r_o^2 - r_i^2),\quad b_1 = r_i\, r_o^2(u_i - r_i)/(r_o^2 - r_i^2), \\ a_2 &= -r_i\, v_i/(r_o^2 - r_i^2),\qquad b_2 = r_i\, r_o^2\, v_i/(r_o^2 - r_i^2), \end{aligned}\right\} \tag{2.3.17}$$

The deformation (2.3.12) with constants (2.3.17) is valid within a certain limit of the prescribed parameters such as ρ_i, τ and the geometrical parameters pertaining to the reference surface Ω. Beyond certain limits, the deformed membrane is partly wrinkled in a fractional annulus $[r_i, \bar{r})$ where $\bar{r}$ is defined by (2.3.15). The tense solution (2.3.12) is valid only in the remaining annulus $(\bar{r}, r_o]$. There are two distinct limiting values for ρ_i and τ corresponding to each one of which a distinct state of incipient wrinkling occurs at the inner boundary r_i. We derive these two limits individually.

(1) Incipient hub rotation τ^{inc} for fixed r_i/r_o

We first consider the case that values of ρ_i and τ are given so that a substantial amount of hub rotation is needed to approach the incipient wrinkling state. The maximum amount of allowable rotation τ may be determined from (2.3.15). We denote this rotation by τ^{inc}. Substitute a_1, a_2, b_1 and b_2 in (2.3.17) into (2.3.15) and setting $\bar{r} = r_i$, the value of τ^{inc} corresponding to such an incipient state must satisfy the following equation

$$2\cos\tau^{inc} = (1 - r_i^2/r_o^2)r_i/\rho_i + (1 + r_i^2/r_o^2)\rho_i/r_i \qquad (2.3.18)$$

where we have used boundary conditions $(2.3.16)_{1,2}$ to replace u_i and u_i by $\rho_i \cos\tau$ and $\rho_i \sin\tau$, respectively.

Clearly, τ^{inc} is dependent on the ratio ρ_i/r_i for the given r_i/r_o. In the range of $0 \le \tau^{inc} \le \pi$, $\cos\tau$ decreases with the increasing τ. For a moderate rotation, a single minimum of $\cos\tau^{inc}$ with respect to ρ_i/r_i may be calculated from (2.3.18) by solving for ρ_i from the algebraic equation $d(\cos\tau^{inc})/d(\rho_i/r_i) = 0$. We denote the corresponding value of ρ_i/r_i by $(\rho_i/r_i)_\tau^{inc}$. Differentiating (2.3.18) and setting it to zero yields a solution for ρ_i/r_i

$$(\rho_i/r_i)_\tau^{inc} = [(1 - r_i^2/r_o^2)/(1 + r_i^2/r_o^2)]^{1/2}. \qquad (2.3.19)$$

This is the value that maximizes the hub-twist angle $|\tau|$ required to arrive at the incipient wrinkling state at the inner boundary. Substitute $(\rho_i/r_i)_\tau^{inc}$ back into (2.3.18) to obtain the corresponding τ^{inc}, denoted by $(\tau^{inc})_{max}$,

$$(\tau^{inc})_{max} = \cos^{-1}\{(1 - r_i^4/r_o^4)^{1/2}\}. \qquad (2.3.20)$$

(2) Incipient hub radius ρ_i^{inc} for $\tau = 0$ and fixed r_i/r_o

We now consider the case in which the state of incipient wrinkling occurs at the inner boundary r_i without hub twist. The values of corresponding ratio (ρ_i/r_i), denoted by $(\rho_i/r_i)^{inc}$ can be obtained by setting $\tau^{inc} = 0$ in equation (2.3.18). Solving for (ρ_i/r_i) from the resulting equation, we obtain two distinct solutions

$$(\rho_i/r_i)_1^{inc} = (1 - r_i^2/r_o^2)/(1 + r_i^2/r_o^2) \quad \text{and} \quad (\rho_i/r_i)_2^{inc} = 1. \qquad (2.3.21)$$

The first of (2.3.21) corresponds to a substantial radial displacement of the membrane before it arrives at the incipient wrinkling without twist. The second states that the hub

radius ρ_i is equal to the inner radius r_i of the undeformed annulus at an incipient wrinkling state, in this case, the entire membrane is wrinkled instantaneously as soon as an infinitesimal amount of twist is applied on the hub. This solution agrees with Reissner and Mansfield results [1.2, 1.3]. For ratios ρ_i/r_i that are less than the first of (2.3.21), the membrane is partly wrinkled as a result of the radial displacement alone. Since the first solution is smaller than the second one, it follows that it is less than 1. For the problems we have considered, ρ_i/r_i is always restricted to the range (0, 1).

Since $\rho_i/r_i \leq 1$, by comparison, solution (2.3.19) can be related to (2.3.21) in the following order

$$(\rho_i/r_i)_1^{inc} \leq (\rho_i/r_i)_\tau^{inc} \leq (\rho_i/r_i)_2^{inc} . \qquad (2.3.22)$$

We have found three values of ρ_i/r_i corresponding to each one of which there is a distinct incipient wrinkling state of equilibrium deformation. We also find that, for a given ratio r_i/r_o, the largest and smallest ratio ρ_i/r_i for the incipient wrinkling are those given by (2.3.21) which entails no hub rotation. For those ρ_i/r_i that satisfy the strict inequality in (2.3.22), a certain amount of hub twist is needed to bring the membrane to the incipient wrinkling state. This amount of twist is given by (2.3.18). The non-monotone behaviour of $\cos\tau$ in this interval can be explained as follows. For $\rho_i/r_i = 1$, wrinkling occurs and spread over the entire membrane instantaneously as soon as $|\tau| = 0^+$ is applied. Likewise, as the limit $(2.3.21)_2$ is approached, the potential wrinkling is imminent and a small rotation of the hub suffices to produce it. Between these limits, substantial rotation is required to cause wrinkling. It can be inferred from (2.3.21) that $\cos\tau$ approaches 1 increasingly as either one of the limit is approached. This implies that

lesser hub rotation is required to initialize wrinkling when values of ρ_i/r_i is near either one of the limits.

For the given r_i/r_o, when the hub-twist angle τ is large than τ^{inc}, or the ratio ρ_i/r_i is smaller than $(\rho_i/r_i)_1^{inc}$, the membrane is partly wrinkled in an inner annular region $[r_i, \bar{r})$ where $r_i < \bar{r} < r_o$. We use the following notations to denote u and v at the boundary $\bar{r}$

$$\bar{u} = u(\bar{r}), \quad \bar{v} = v(\bar{r}). \tag{2.3.23}$$

The deformation corresponding to the tense region $(\bar{r}, r_o]$ is determined by (2.3.12) in which the constant a_1, a_2, b_1 and b_2 are determined by considering boundary conditions $(2.3.16)_{3,4}$ and (2.3.23):

$$\left.\begin{aligned} a_1 &= (r_o^2 - \bar{u}\bar{r})/(r_o^2 - \bar{r}^2), \quad b_1 = \bar{r}r_o^2\,(\bar{u}-\bar{r})/(r_o^2 - \bar{r}^2), \\ a_2 &= -\bar{r}\bar{v}/(r_o^2 - \bar{r}^2), \qquad b_2 = \bar{r}r_o^2\,\bar{v}/(r_o^2 - \bar{r}^2). \end{aligned}\right\} \tag{2.3.24}$$

in which values of parameters $\bar{r}$, $\bar{u}$ and $\bar{v}$ are not known *a priori* and are to be determined by imposing continuity conditions at $\bar{r}$. Along the circle $\bar{r}$, solutions of the tense region and the wrinkled region must be matched according to certain continuity conditions. Determination of $\bar{r}$, $\bar{u}$ and $\bar{v}$ constitutes part of the problem.

To maintain the equilibrium of the deformation with twist, a certain amount of torque must be applied on the hub. To calculate its magnitude, we first calculate the traction transmitted across a circle $\tilde{r}$ = const. by the material in the annulus $(\bar{r}, r_o]$. The unit exterior normal vector to the outer boundary of the isolated body of the circular disc with radius $\tilde{r}$ is $\underline{i}(\theta)$.

For axisymmetric deformations, deformation gradient $\underline{F}$, its inverse $\underline{F}^{-1}$ and transpose

of the inverse $\underline{F}^{-T}$ are given in (2.1.7, 8, 9), respectively. Substituting these expressions into equation (2.2.13) to obtain the Piola stress tensor $\underline{T}$, we obtain the traction $\underline{t} = \underline{T}\underline{i}(\theta)$

$$(2G)^{-1}\underline{t} = I^{-1}\, dF/dI\, [(u'+u/r)\underline{j}(\theta) + (v'+v/r)\underline{j}(\theta)] - r^{-1}\underline{y}, \qquad (2.3.25)$$

where $\underline{y}$ is given by (2.1.5).

Integration of the cross product of $\underline{y} \times \underline{t}$ around the circle $\bar{r}$ yields the total torque $\underline{M}$:

$$\underline{M} = \int_0^{2\pi} \underline{y} \times \underline{t}\, d\theta. \qquad (2.3.26)$$

using the deformation (2.3.12) in (2.1.5) and substituting it into (2.3.26), we find that $\underline{M} = M\underline{k}$, where

$$M = 8\pi G I^{-1}\, dF/dI\, (a_2 b_1 - a_1 b_2). \qquad (2.3.27)$$

M is independent of the choice of $\bar{r}$, as expected from elementary equilibrium consideration. The torque transmitted to the membrane by the hub is equal to M in magnitude, but has the opposite sign. Constants a_1, a_2, b_1 and b_2 will be determined after the solution is obtained.

2.4 Solution in the wrinkled region

As indicated in the later part of the forgoing section, hub radius ρ_i and twist angle τ can be specified such that part of the deformed membrane is wrinkled. When such a deformation occurs, the membrane is wrinkled in the annulus $[r_i,\bar{r})$. The stress field in the wrinkled region is referred to as the tension field. In the region $[r_i,\bar{r})$, we assume that $\lambda > 1$ and $\mu \leq \upsilon(\lambda)$, $\upsilon(\lambda)$ is the natural width defined by $(2.2.3)_2$ for the special harmonic material. This assumption will be verified in section 2.7. We invoke the

tension-field theory to establish the equations of equilibrium for the deformation in the annular region $r \in [r_i, \bar{r})$. Deformations for the tense region $(\bar{r}, r_o]$ have been obtained in section 2.3 and they are presented in the general form (2.3.12). The constants are determined in (2.3.24).

In general, equilibrium equations for the wrinkled parts can be obtained directly using the minimum energy principal for the potential energy (2.3.1) in which the integrand $H(\cdot)$ is replaced by the relaxed strain energy density $W^R(\lambda)$. By definition, $W^R(\lambda)$ is independent of μ in the tension field. Hence, in view of the expression of λ in $(2.1.16)_1$ for the current problem, $W^R(\lambda)$ depends on r explicitly. The resulting Euler-Lagrange equations are rather complex and it is difficult to solve them analytically. However, for the problem in which no lateral pressure is applied on the membrane, we are able to take advantage of some special features of the tension field theory to construct solutions for the wrinkled regions.

Substituting $(2.1.13)_1$ into the equilibrium equation $(1.4.7)_1$ and noting that $\underline{k} = \underline{N}$, we obtain two equations of the form

$$(rf\cos\gamma)_r + (f\sin\gamma)_\theta = 0, \quad (r\lambda\sin\gamma)_r - (\lambda\cos\gamma)_\theta = 0, \tag{2.4.1}$$

Equation (2.1.16, 17) implies that λ and γ are independent of θ. It follows that the second terms on the left side of (2.4.1) vanish identically. Equation (2.4.1) can be integrated once with respect to r to yield

$$f(\lambda)\cos\gamma = a/r \quad \text{and} \quad \lambda\sin\gamma = b/r, \tag{2.4.2}$$

where a and b are integration constants to be determined by properly imposed boundary conditions. To write r and γ in terms of λ, we square, add and divide $(2.4.2)_{1,2}$ to get

$$r^2 = [a/f(\lambda)]^2 + (b/\lambda)^2, \quad \tan\gamma = \lambda^{-1}f(\lambda)b/a. \tag{2.4.3}$$

Equation (2.4.3) were first derived by Wu and Canfield [1.6] without the benefit of the general theory developed in section 1.4.

Equation $(2.4.3)_1$ may be used profitably as a relation by which functions of r can be converted into functions of λ. Owing to the nonlinear strain-stress relation $f(\lambda)$ (2.2.3), such conversion is valid if and only if the relation between r and λ is one-to-one. To prove this, we first calculate the derivative of r with respect to λ. It is

$$\frac{dr}{d\lambda} = -\frac{1}{r}\left[\frac{a^2}{f(\lambda)^3}\, f'(\lambda) + \frac{b^2}{\lambda^3}\right] \tag{2.4.4}$$

From equations (2.2.3), it shows that $f(\lambda) > 0$, $f'(\lambda) > 0$, for $\lambda > 1$. Then the value in the square bracket is strictly positive. This leads to the conclusion that $dr/d\lambda$ is strictly negative. If $r > 0$, $dr/d\lambda$ is negative at every point r in the annulus $[r_i,\bar{r})$. Hence, r is a strictly decreasing function of λ for any fixed values *a* and *b*. Conversely, λ in $(2.3.1)_1$ is also uniquely determined as a monotone decreasing function of r if $\lambda > 1$. Because this monotonic dependence of λ on r is given implicitly, it is convenient to use λ instead of r as the independent variable. The orientation of the principal vectors $\{\underline{L},\underline{M}\}$ of the strain tensor $\underline{C}$ in the reference surface Ω is determined by the angle γ calculated from $(2.4.3)_2$ and it is an explicit function of λ. We conclude this paragraph by emphasizing that we are going to use λ as the independent variable. Functions of r and γ can be converted into functions of λ via the use of equations $(2.4.3)_{1,2}$.

In the absence of lateral pressure ($p^d = 0$), the deformed membrane in wrinkled annulus $[r_i,\bar{r})$ is a ruled surface. The general solution for such a surface is presented in

equation (1.4.23) in which the deformation $\underline{y}$ is expressed in terms of the curvilinear coordinates $\{\phi,\psi\}$. To particularize the general deformation appropriately to the current problem, we need the functional forms of ϕ and ψ in terms of r, or equivalently, λ. By definition, ϕ and ψ are solutions of equations (1.4.9) and (1.4.12) which are nearly arbitrary subject only to condition (1.4.14). By substituting $(2.1.13)_{1,2}$ into (1.4.9, 12), we have

$$\cos\gamma\phi_r + r^{-1}\sin\gamma\phi_\theta = \lambda, \quad \cos\gamma\psi_r + r^{-1}\sin\gamma\psi_\theta = 0. \tag{2.4.5}$$

These are two independent partial differential equations for ϕ and ψ. In axisymmetrical deformation, γ is the function of r only. Apparently, solutions of $\{\phi,\psi\}$ are related to the deformations through their dependence on γ. Thus the coordinate net formed by $\{\phi,\psi\}$ in the reference annulus $[r_i,\bar{r})$ changes with deformations. By contrast, coordinate $\{r,\theta\}$ is fixed in Ω, however.

To solve equations (2.4.5) for ϕ and ψ, we inspect that the second of (2.4.5) determines the ratio of the derivatives of ψ. Particularly, this equation can be satisfied if we take

$$\psi_\theta = 1, \quad \text{and } \psi_r = -\, r^{-1}\tan\gamma. \tag{2.4.6}$$

As a result, integration of $(2.4.6)_1$ gives

$$\psi = \theta + \delta(r). \tag{2.4.7}$$

To determine the unknown function $\delta(r)$, we differentiate (2.4.7) with respect to r to obtain $\psi_r = \delta'(r)$ and compare it with $(2.4.5)_2$ to derive

$$\delta'(r) = -r^{-1}\tan\gamma. \tag{2.4.8}$$

To determine ϕ, we can let $\phi_\theta = 0$ in $(2.4.5)_1$. Moreover, taking $\phi = 0$ and $\psi = \theta$

at $r = r_i$, we obtain ϕ and ψ in terms of r as

$$\phi(r) = \int_{r_i}^{r} \lambda \sec\gamma \, dx \qquad (2.4.9)$$
$$\phi_r = \lambda\sec\gamma; \qquad \phi_\theta = 0$$

and

$$\psi = \theta - \delta(r) \; ;$$
$$\delta(r) = \int_{r_i}^{r} x^{-1} \tan\gamma \, dx \, . \qquad (2.4.10)$$

We can convert ϕ and δ to functions of λ by using equation (2.4.3) in which $f(\lambda)$ is given by $(2.2.3)_1$. (see Appendix A).

The function $\phi(r)$ measures the deformed length of the tension lines ψ = const. in the wrinkled annulus $[r_i,\bar{r})$. It is evident from $(2.4.2)_1$ that the integrand of $(2.4.9)_1$ $\lambda\sec\gamma = r\lambda f(\lambda)/a$ unless $a = 0$. Moreover, the integrand itself is positive or negative in $[r_i,\bar{r})$ if and only if a is positive or negative. We assume $a > 0$ such that the length of tension lines is a positive increasing function of radius r. If $a = 0$, then, from $(2.4.2)_1$ $\cos\gamma = 0$. Then relation $(2.1.13)_1$ implies that $\underline{L}$ is purely azimuthal: $\underline{L} = \pm\underline{j}(\theta)$. In this case, the tension lines form a family of concentric circles. Since $\underline{L} \cdot \underline{i}(\theta) = 0$, (1.4.3) delivers no traction across the circle with the normal $\underline{i}(\theta)$, and the torque required to twist the membrane and the force required to deform it radially would vanish identically. We exclude this deformation from our consideration, and thus $a > 0$ is assumed.

We use (2.4.9, 10) and the equilibrium equation (2.4.2) to calculate

$$\underline{\nabla}\phi = \phi'(r)\underline{\nabla}r = a^{-1}r\lambda f(\lambda)\underline{i}(\theta),$$

$$\underline{\nabla}\psi = \underline{\nabla}\theta - \delta'(r)\underline{\nabla}r = r^{-1}[\underline{j}(\theta) - \tan\gamma\underline{i}(\theta)]. \tag{2.4.11}$$

where $\underline{\nabla}(\,) = (\,)_{,r}\,\underline{i}(\theta) + r^{-1}(\,)_{,\theta}\,\underline{j}(\theta)$.

According to (1.4.14)

$$k\cdot\nabla\phi \times \nabla\psi = \lambda f(\lambda)/a. \tag{2.4.12}$$

Since $a > 0$ as we assumed previously, (2.4.12) is strictly positive for $\lambda > 1$ at each point $r \in [r_i,\bar{r})$, so the assumption of section 1.4 that ϕ and ψ are functionally independent is verified.

It remains to determine the functional form of $\underline{u}$ in the general form of the deformation (1.4.23). Using $(2.4.9)_2$ and (2.4.10), the derivatives of $\underline{y}(\underline{x})$ with respect to r, θ are

$$\underline{y}_{,r} = \underline{y}_{,\phi}\phi_{,r} + \underline{y}_{,\psi}\psi_{,r} = \lambda\sec\gamma\underline{l} - r^{-1}\tan\gamma(\phi\underline{l}_{,\psi} + \underline{u}_{,\psi});$$

$$\underline{y}_{,\theta} = \underline{y}_{,\phi}\phi_{,\theta} + \underline{y}_{,\psi}\psi_{,\theta} = \phi\underline{l}_{,\psi} + \underline{u}_{,\psi}. \tag{2.4.12}$$

From $(1.2.23)_2$ and $(2.1.13)_4$, we have the relation

$$\mu\underline{m} = -\sin\gamma\underline{y}_{,r} + r^{-1}\cos\gamma\underline{y}_{,\theta}. \tag{2.4.13}$$

Substituting $(2.4.12)_{1,2}$ into the above equation, it becomes

$$\mu\underline{m} = -\lambda\tan\gamma\underline{l} + r^{-1}\sec\gamma(\phi\underline{l}_{,\psi} + \underline{u}_{,\psi}) \tag{2.4.14}$$

Since $\underline{l}$ is a vector with unit magnitude, it is orthogonal to its derivative, $\underline{l}\cdot\underline{l}_{,\psi} = 0$. Thus the orthogonality condition $\underline{l}\cdot\underline{m} = 0$ requires (2.4.14) to satisfy

$$\underline{l}\cdot\underline{u}_{,\psi} = r\lambda\sin\gamma = b, \tag{2.4.15}$$

where b is the constant in $(2.4.2)_2$.

Recall that $\phi = 0$, $\psi = \theta$ at $r = r_i$, from (1.4.23), the deformation vector $\underline{y}$ at r_i furnishes

$$\underline{y}(r_i,\theta) = \underline{u}(\theta). \tag{2.4.16}$$

This must be equal to the displacement boundary conditions prescribed in (2.3.16) at the inner boundary r_i. Hence, it can be written in the form

$$\underline{u}(\theta) = u_i\underline{i}(\theta) + v_i\underline{j}(\theta) = \rho_i\underline{i}(\theta + \tau), \tag{2.4.17}$$

where

$$\underline{i}(\theta + \tau) = \cos\tau\underline{i}(\theta) + \sin\tau\underline{j}(\theta), \quad \underline{j}(\theta + \tau) = -\sin\tau\underline{i}(\theta) + \cos\tau\underline{j}(\theta). \tag{2.4.18}$$

Since $\underline{u}$ is the deformation vector at the inner boundary r_i, from (2.4.17), we infer that $\underline{u}$ has the form

$$\underline{u}(\psi) = u_i\underline{i}(\psi) + v_i\underline{j}(\psi) = \rho_i\underline{i}(\psi + \tau), \tag{2.4.19}$$

where $\underline{j}(\psi) = \underline{i}'(\psi)$, and $\underline{i}(\psi + \tau)$ is obtained from $(2.4.18)_1$ in which θ is replaced by ψ. $\underline{i}(\psi + \tau)$ and $\underline{j}(\psi + \tau)$ can be resolved in the base $\{\underline{i}(\psi),\underline{j}(\psi)\}$ in the form that are similar to equation (2.4.18), i.e.

$$\underline{i}(\psi + \tau) = \cos\tau\underline{i}(\psi) + \sin\tau\underline{j}(\psi), \quad \underline{j}(\psi + \tau) = -\sin\tau\underline{i}(\psi) + \cos\tau\underline{j}(\psi). \tag{2.4.20}$$

On the deformed surface, we define a local angle β which is formed between the principal vector $\underline{l}(\psi)$ and the radial vector $\underline{i}(\psi)$ such that

$$\underline{l} = \cos\beta\underline{i}(\psi) + \sin\beta\underline{j}(\psi), \quad \underline{m} = -\sin\beta\underline{i}(\psi) + \cos\beta\underline{j}(\psi). \tag{2.4.21}$$

Note that, in deformations involving twist, β is not the same as the angle α defined in (2.1.13) except at $r = r_i$. β equals to α identically if the twist is zero and the deformation is purely radial.

To interpret β in an appropriate way, we differentiate $\underline{u}$ in (2.4.19) with respect to ψ and scalar-multiplying it by $\underline{l}$, we obtain the relation $b/\rho_i = \underline{l}\cdot\underline{j}(\psi + \tau)$. If $\underline{l}$ and $\underline{j}(\psi + \tau)$ are replaced by equations $(2.4.21)_1$ and (2.4.20), respectively, the relation becomes

$$b/\rho_i = \sin(\beta - \tau). \tag{2.4.22}$$

When ρ_i and τ are prescribed, β is a constant at radius r. This implies that tension lines on the deformed surface have a constant angle with radial vector $\underline{i}(\psi) = \underline{i}(\theta - \delta(r))$. A fixed radial ray $\underline{i}(c)$, c = const. on the deformed surface corresponds to the curve $\theta + \Gamma(r) = c$ on the reference surface Ω. On this curve, $\psi = \theta - \delta(r) = c - \Gamma(r) - \delta(r)$ is a particular function of r. It can also be expressed as some other function of the deformed radius ρ. It may be seen, as shown in Fig. 2.4.1, that as ρ progresses along the ray, different tension lines are encountered at a variable angle β to the ray.

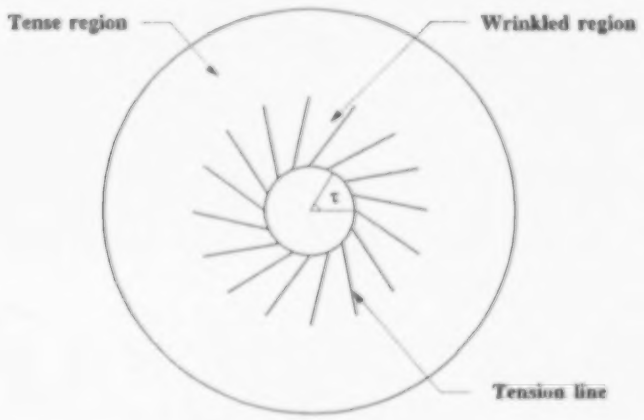

Fig. 2.4.1 Partly wrinkled membrane

The foregoing results can be used to relate the integration constants a, b to the torque required to maintain the deformation satisfying equilibrium equations (2.4.2). The constant a has certain physical interpretation. From (1.4.3), $(2.1.13)_1$ and $(2.4.2)_1$, the traction exerted by the material in the annulus $[r_i, \tilde{r})$ across the circle $\tilde{r}$ = const. is

$$\underline{t} = f(\lambda)\cos\gamma\ \underline{l} = (a/\tilde{r})\ \underline{l}. \qquad (2.4.23)$$

where $r_i < \bar{r} < i$. The constant a can be interpreted as the constant total force channelled between two radial rays $\underline{i}(\theta)$ and $\underline{i}(\theta + \Delta\theta)$.

Torque generated by the traction $\underline{t}$ per unit circumferential length about the axis of symmetry is calculated by the cross-product of (1.4.23) and (2.4.23) in which $\underline{t}$ and $\underline{l}$ are replaced by (2.4.19) and (2.4.21), respectively. It is

$$\underline{y} \times \underline{t} = (a/\bar{r})\underline{u} \times \underline{l} = (a/\bar{r})\, \rho_i \sin(\beta - \tau)\, \underline{k}. \tag{2.4.24}$$

The total torque is the closed-path integration of (2.4.24) around the reference circle $\bar{r}$ = const.. Substituting (2.4.22) into (2.4.24), the total torque is found to be

$$\underline{M} = 2\pi ab\, \underline{k}. \tag{2.4.25}$$

which is independent of the reference circle $\bar{r}$ along which it is integrated. This is required by equilibrium.

As noted in section 2.3, the torque exerted by the hub on the membrane is $-\underline{M}$. We take the hub rotation to be counterclockwise (Fig. 2.4.1). To obtain a torque of the same sense, we assume that $b \leq 0$, since we have already assumed a to be positive.

To match the wrinkled solution (1.4.23) with the fully tense solution (2.3.12) at the interior boundary $r = \bar{r}$, it is convenient to write $\underline{y}$ in (1.4.23) into the comparable form as those in (2.3.12). To achieve this, we substitute $(2.4.19)_1$ and $(2.4.21)_1$ into (1.4.23) to obtain

$$\underline{y} = (\phi\cos\beta + \rho_i \cos\tau)\underline{i}(\psi) + (\phi\sin\beta + \rho_i \sin\tau)\underline{j}(\psi). \tag{2.4.26}$$

Furthermore, $\underline{y}$ may be expressed in terms of the vectors $\underline{i}(\theta)$ and $\underline{j}(\theta)$ if $\{\underline{i}(\psi),\underline{j}(\psi)\}$ are resolved in the base $\{\underline{i}(\theta), \underline{j}(\theta)\}$. Because $\psi = \theta - \delta(r)$, in view of (2.4.20), we have

$$\underline{i}(\psi) = \underline{i}(\theta - \delta(r)) = \cos\delta(r)\underline{i}(\theta) - \sin\delta(r)\underline{j}(\theta),$$

$$\underline{j}(\psi) = \underline{j}(\theta - \delta(r)) = \sin\delta(r)\underline{i}(\theta) + \cos\delta(r)\underline{j}(\theta). \tag{2.4.27}$$

Using (2.4.27) in (2.1.13), principal vectors $\underline{l}$ and $\underline{m}$ can be written as

$$\left.\begin{aligned} \underline{l} &= \cos(\beta-\delta)\underline{i}(\theta) + \sin(\beta-\delta)\underline{j}(\theta), \\ \underline{m} &= \sin(\delta-\beta)\underline{i}(\theta) + \cos(\delta-\beta)\underline{j}(\theta). \end{aligned}\right\} \tag{2.4.28}$$

Substituting (2.4.27) into (2.4.26) and comparing it with $\underline{y}$ in (2.1.5) obtains

$$\left.\begin{aligned} u(r) &= \phi\cos(\beta-\delta) + \rho_1\cos(\tau-\delta) \\ v(r) &= \phi\sin(\beta-\delta) + \rho_1\sin(\tau-\delta) \end{aligned}\right\}, \tag{2.4.29}$$

To convert u and v into functions of λ, we assume that the deformed surface is smooth with class C^1. This condition requires that the deformation gradient $\underline{F}$ is continuous on the reference surface Ω. Therefore, $\underline{F}$ expressed in (1.2.22) and (2.1.7) must be equal. Moreover, continuity of $\underline{F}$ at a point implies that any differential material vector passing through that point in the reference surface will be mapped into the corresponding differential material vector on the deformed surface. In particular, this implies the equality of $\underline{F}\underline{i}(\theta)$. From (1.2.22) and (2.1.7), this is

$$u'(r)\underline{i}(\theta) + v'(r)\underline{j}(\theta) = \lambda\underline{l}\cos\gamma - \mu\underline{m}\sin\gamma, \tag{2.4.30}$$

where $(2.1.13)_{1,2}$ have been used for $\underline{L}$, $\underline{M}$. Substituting (2.4.21) and (2.4.27) to eliminate $\underline{l}$ and $\underline{m}$ and recall that vectors $\underline{i}(\theta)$ and $\underline{j}(\theta)$ are linearly independent, we derive

$$\left.\begin{aligned} u'(r) &= \gamma\cos\gamma\cos(\beta - \delta) + \mu\sin\gamma\sin(\beta - \delta) \\ v'(r) &= \gamma\cos\gamma\sin(\beta - \delta) + \mu\sin\gamma\cos(\beta - \delta) \end{aligned}\right\}, \tag{2.4.31}$$

Also, equality of $\underline{F}\underline{j}(\theta)$ yields the equation

$$-(v/r)\underline{i}(\theta) + (u/r)\underline{j}(\theta) = \lambda\underline{l}\sin\gamma + \mu\underline{m}\cos\gamma. \tag{2.4.32}$$

Substituting (2.4.21) and (2.4.27) into the right-hand side of above equation and

grouping the respective coefficients of $\mathbf{j}(\theta)$ and $\mathbf{j}(\theta)$, we get

$$\left.\begin{aligned} u/r &= \lambda \sin\gamma \sin(\beta-\delta) + \mu \cos\gamma \cos(\delta-\beta) \\ v/r &= -\{\lambda \sin\gamma \cos(\beta-\delta) + \mu \cos\gamma \sin(\delta-\beta)\} \end{aligned}\right\} \qquad (2.4.33)$$

Squaring and adding $(2.4.33)_{1,2}$, together with (2.4.2) to eliminate $\sin\gamma$ and $\cos\gamma$, we arrive at the equation

$$u^2 + v^2 = (a\mu/f(\lambda))^2 + b^2. \qquad (2.4.34)$$

Substituting (2.3.34) into the right-hand side of $(2.1.11)_b$ and using $(1.2.26)_2$ to eliminate μ, we have

$$d(\mu/f(\lambda))/dr = r\lambda f(\lambda)/a^2 > 0, \ \lambda > 1, \ r \in [r_i,\bar{r}). \qquad (2.4.35)$$

Positiveness of derivative (2.4.35) indicates that $\mu/f(\lambda)$ is a monotone increasing function of r. Since $f(\lambda) > 0$ for $\lambda > 1$, the present solution is therefore limited by the condition $\mu = 0$. According to equation (2.4.34), $\mu = 0$ can only occur at the inner boundary r_i where modulus of the deformed radius $(u^2 + v^2)^{1/2}$ takes its minimum. Recall the assumption $b \leq 0$, it follows that at this state $b = -\rho_i$. Thus $\mu(r_i) = 0$ is the least state that the current solution is valid. The state of $\mu(r_i) = 0$ is interpreted as a local collapse of the material at the inner boundary r_i (membrane is squashed in the direction of μ at this state of strain). Because the membrane is wrinkled in the neighbourhood of r_i, elastic behaviour of the membrane is dictated by the constitutive relation derived from the relaxed strain energy $w^R(\lambda)$ which is independent of μ. Thus, the collapse of the material does not change the local strain energy. For the hub with definite radius, i.e. $\rho_i \neq 0$, $\mu(r_i) = 0$ is unlikely in purely radial displacements. Therefore, in any case, such material collapse always involves a certain amount of twist on the hub. We denote

this twist by τ_{max}. To calculate τ_{max}, we use the condition $b = -\rho_i$ in (2.4.22) to derive the limiting condition

$$\tau_{max} - \beta = \pi/2. \tag{2.4.36}$$

To interpret this limiting condition physically, we rewrite (2.4.36) in the form

$$(\tau_{max} - \delta) + (\delta - \beta) = \pi/2, \tag{2.4.37}$$

where $\delta(r)$ is defined in $(2.4.10)_2$. Identity of (2.4.37) is equivalent to the condition:

$$\cos(\delta - \tau_{max}) \cos(\delta - \beta) + \sin(\delta - \tau_{max}) \sin(\delta - \beta) = 0. \tag{2.4.38}$$

By means of (2.4.27) and $(2.3.16)_{1,2}$, $\underline{u}(\psi)$ in (2.4.19) can be expressed as

$$\underline{u}(\psi) = \rho_i \cos(\tau - \delta)\, \underline{i}(\theta) + \rho_i \sin(\tau - \delta)\, \underline{j}(\theta). \tag{2.4.39}$$

Setting $\tau = \tau_{max}$ in (2.4.39) and scalar-multiplying it with $\underline{l}$ in $(2.4.28)_1$, according to condition (2.4.38), the product is

$$\underline{u} \cdot \underline{l} = 0, \quad \text{at } r = r_i,\ \tau = \tau_{max} \tag{2.4.40}$$

Because $\underline{l}(r_i)$ is tangential to a tension line on the deformed surface, $\underline{u}(r_i)$ is the position vector of the point r_i attached to the hub, and orthogonality of $\underline{l}$ and $\underline{u}$ at r_i implies that tension line is tangential to the hub in the limiting deformation.

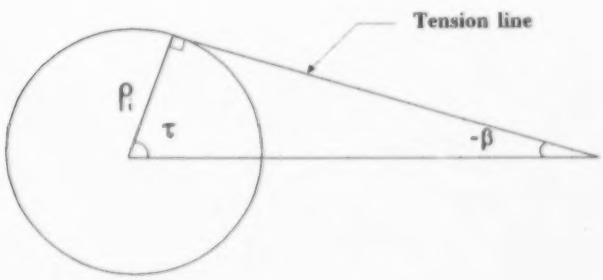

Fig. 2.4.2 Tension lines are tangential to the hub in the limiting state $\tau = \tau_{max}$

The limiting deformation is illustrated in Fig. 2.4.2. For $\tau > \tau_{max}$ we expect that tension lines will wrap around the hub. Our solution does not account for this, however.

2.5 Matching condition

In section 2.3 and 2.4, we have assumed that deformation $\underline{y}(\underline{x})$ and its gradient $\underline{F}$ are continuous in the individual tense and wrinkled parts on the deformed surface. We further assume that $\underline{y}(\underline{x})$ and $\underline{F}(\underline{x})$ are continuous across the common boundary $\bar{r}$. This assumption causes no essential loss of generality. From the solutions we have obtained, the interior boundary $\bar{r}$ is a circle which intersects tension lines at a non-zero angle. Therefore $\underline{F}$ must be necessarily continuous at $\bar{r}$. Continuity of $\underline{F}$ requires that $u(r)$, $v(r)$ and $u'(r)$, $v'(r)$ be continuous at $r = \bar{r}$.

In accordance with the notation $\bar{r}$, we use overbars to denote the values of functions evaluated at $r = \bar{r}$. By definition, $\bar{\mu} = \upsilon(\bar{\lambda})$ at $\bar{r}$. Continuity of $\underline{F}$ at $\bar{r}$ requires that u' and v' derived from solution (2.3.12) for the tense region must be equal to those in the wrinkled region derived in (2.4.31), thus, we have

$$\left.\begin{aligned} \bar{\lambda}\cos\bar{\gamma}\cos(\beta - \bar{\delta}) + \bar{\upsilon}\sin\bar{\gamma}\sin(\beta - \bar{\delta}) &= a_1 - b_1/\bar{r}^2 \\ \bar{\lambda}\cos\bar{\gamma}\sin(\beta - \bar{\delta}) + \bar{\upsilon}\sin\bar{\gamma}\cos(\beta - \bar{\delta}) &= a_2 - b_2/\bar{r}^2 \end{aligned}\right\}, \qquad (2.5.1)$$

where $\bar{\upsilon} = \upsilon(\bar{\lambda})$, a_1, b_1, a_2, b_2 are the integration constants listed in (2.3.24). They are dependent on $\bar{u}$ and $\bar{v}$. Evaluating (2.4.29) at $\bar{r}$, we have:

$$\bar{u} = \bar{\phi}\cos(\beta-\bar{\delta}) + \rho_i\cos(\tau-\bar{\delta}), \quad \bar{v} = \bar{\phi}\sin(\beta-\bar{\delta}) + \rho_i\sin(\tau-\bar{\delta}), \qquad (2.5.2)$$

Using (2.3.24) to calculate the right-hand side of (2.5.1), they are expressed in terms of the data at $\bar{r}$ by the equations

$$a_1 - b_1/\bar{r}^2 = 2A_2 - A_1\bar{u}, \quad a_2 - b_2/\bar{r}^2 = -A_1\bar{v}, \tag{2.5.3}$$

where

$$A_1 = \bar{r}^{-1}(1 + \bar{r}^2/r_o^2)A_2 \text{ and } A_2 = (1 - \bar{r}^2/r_o^2)^{-1}. \tag{2.5.4}$$

To write equations (2.5.1) in a concise form, we first multiply $(2.5.1)_1$ by $\cos(\beta-\bar{\delta})$, $(2.5.1)_2$ by $\sin(\beta-\bar{\delta})$, and add the results. Next, we multiply $(2.5.1)_2$ by $\cos(\beta-\bar{\delta})$ and subtract the result from the multiplication of $\sin(\beta-\bar{\delta})$ by $(2.5.1)_1$. Using (2.5.2) and (2.5.3) in the resulting equations, we obtain

$$\left.\begin{aligned} \bar{\lambda}\cos\bar{\gamma} + A_1[\bar{\phi} + \rho_i\cos(\beta-\tau)] &= 2A_2\cos(\beta-\bar{\delta}) \\ \upsilon(\bar{\lambda})\sin\bar{\gamma} + A_1\rho_i\sin(\beta-\tau) &= 2A_2\sin(\beta-\bar{\delta}) \end{aligned}\right\}, \tag{2.5.5}$$

We need one more equation to from a close-system of equations for three unknowns β, λ_i and $\bar{\lambda}$. To this end, following the substitution of $(2.5.2)_{1,2}$ into (2.4.34), we obtain the equation

$$\bar{\phi}^2 + 2\bar{\phi}\rho_i\cos(\beta-\tau) + \rho_i^2 = (a\upsilon(\bar{\lambda})/f(\bar{\lambda}))^2 + b^2. \tag{2.5.6}$$

Using (2.4.22) to replace b, it becomes

$$[\bar{\phi} + \rho_i\cos(\beta-\tau)]^2 = (a\upsilon/f(\bar{\lambda}))^2. \tag{2.5.7}$$

For the problem involving no twist, $\beta = \tau = 0$, $\cos(\beta-\tau) = 1$, the term in the square brackets is $\bar{\phi} + \rho_i$ corresponding to the radius of the common boundary circle $\bar{r}$. In the state of maximum twist defined by (2.4.36), $\cos(\beta-\tau) = 0$, this term becomes $\bar{\phi}$ alone which represents the total length of a tension line in the wrinkled region measured from r_i to $\bar{r}$. Both values of this term are associated with the positive square root of (2.5.7). For intermediate twist angles, $0 < \tau - \beta < \pi/2$ (Fig. 2.2.2), $\cos(\tau-\beta)$ is positive, therefore the positive root continues to apply. Hence we choose the positive root

$$\bar{\phi} + \rho_i \cos(\beta - \tau) = a\bar{g}/\bar{f}. \tag{2.5.8}$$

To use a in (2.5.8), it must be expressed as the function of β, λ_i and $\bar{\lambda}$. This is accomplished by evaluating $(2.4.2)_1$ at r_i in which b is substituted by (2.4.22), we get

$$a = \rho_i \lambda^{-1}_i f(\lambda_i)[\lambda_i^2 (r_i/\rho_i)^2 - \sin^2(\beta - \tau)]^{1/2}. \tag{2.5.9}$$

Equations (2.5.3, 4) contain the variable $\bar{r}$ which needs to be expressed as a function of β, λ_i and $\bar{\lambda}$. This is performed by substituting (2.4.22) into $(2.4.3)_1$ and evaluating the equation at $\bar{r}$, it is

$$\bar{r} = \{ [a/f(\bar{\lambda})]^2 + (\rho_i \sin(\beta-\tau)/\bar{\lambda})^2 \}^{1/2} \tag{2.5.10}$$

in which a is replaced by (2.5.9). Following the previous reasoning, we take positive square root for $\bar{r}$. It remains to have $\cos\bar{\gamma}$ and $\sin\bar{\gamma}$ in (2.5.5) be expressed in terms of β, λ_i and $\bar{\lambda}$. Evaluating equations (2.4.2) at $\bar{r}$, they are

$$\cos\bar{\gamma} = a / \bar{r}\, f(\bar{\lambda}); \quad \sin\bar{\gamma} = b / \bar{r}\, \bar{\lambda} \tag{2.5.11}$$

Moreover, (A.4) and (A.11) of Appendix A give $\bar{\phi}$ and $\bar{\delta}$ as functions of these same three parameters. Collecting these various functions (2.5.9, 10, 11) and (A.4) and (A.11) into equations (2.5.5) and (2.5.8), we obtain a system of three highly nonlinear algebraic equations in three variables β, λ_i and $\bar{\lambda}$. A similar system of three coupled nonlinear equations was derived by Mikulas [2.5] in a study of the infinitesimal-deformation problem.

Solutions of β, λ_i and $\bar{\lambda}$ determine the constants $\bar{u}$ and $\bar{v}$ in (2.5.2), which in turn can be substituted into (2.3.24) to obtain the deformation (2.3.12) in the tense part of the membrane. Displacements in the wrinkled region are given by equations (2.4.29) in which u, v are presented as functions of $\lambda \in [\bar{\lambda}, \lambda_i]$. Because the relation between λ and

r is one-to-one, the entire solution can, in principle, be specified as a function of the radius r for fixed values of r_i, r_o and the prescribed ρ_i, τ.

2.6 Non-dimensionalization

To generalize above solutions to annuli with arbitrarily given r_i and r_o, it is necessary to employ nondimensional quantities. To reduce the system of equations $(2.5.5)_{1,2}$ and (2.5.8) into the non-dimensional form, we introduce the following dimensionless quantities:

$$\left.\begin{aligned} &f^* \equiv \bar{f}/G = 2(\bar{\lambda} - \bar{\lambda}^{-1}), \qquad b^* \equiv b/\rho_i = \sin(\beta - \tau), \\ &a^* \equiv \rho_i^{-1} a/G = 2(1-\lambda_i^{-2})\left[(r_i/\rho_i)^2 \lambda_i^2 - (b^*)^2\right]^{\frac{1}{2}}, \\ &A_3 = \bar{\lambda}\left[(a^*\bar{\lambda})^2 + (b^* f^*)^2\right]^{-\frac{1}{2}} \end{aligned}\right\} \qquad (2.6.1)$$

Using these in formulae $(2.4.2)_{1,2}$ and $(2.4.3)_1$, we obtain

$$r^* \equiv \bar{r}/\rho_i = (f^* A_3)^{-1}, \quad \cos\bar{\gamma} = a^* A_3, \quad \sin\bar{\gamma} = \bar{\lambda}^{-1} b^* f^* A_3. \qquad (2.6.2)$$

From (2.5.4) we have $A_1 = \rho_i^{-1} A_1^*$, where

$$A_1^* = (r^*)^{-1}[1 + (r^*)^2(\rho_i/r_0)^2]A_2, \quad A_2 = [1 - (r^*)^2(\rho_i/r_0)^2]^{-1}. \qquad (2.6.3)$$

We also define $\phi^* = \phi(\bar{\lambda})/\rho_i$. This can be obtained from (A.4) by substituting a^* in place of $\rho_i^{-1}a/G$ and $(b^*)^2/a^*$ in place of $\rho_i^{-1}Gb^2/a$. Similarly, $\bar{\delta} = \delta(\bar{\lambda})$ follows from (A.12) after replacement of Gb/a by b^*/a^*. Then $(2.5.5)_{1,2}$ and (2.5.8) furnish three equations for β, λ_i and $\bar{\lambda}$:

$$\left.\begin{aligned} &\bar{\lambda} a^* A_3 + A_1^*[\phi^* + \cos(\beta - \tau)] = 2A_2\cos(\beta - \bar{\delta}), \\ &(\bar{\lambda}^{-2} f^* A_3 + A_1^*)\sin(\beta - \tau) = 2A_2\sin(\beta - \bar{\delta}), \\ &\phi^* + \cos(\beta - \tau) = a^*(\bar{\lambda} f^*) - 1. \end{aligned}\right\} \qquad (2.6.4)$$

in which the condition $v(\bar{\lambda}) = \bar{\lambda}^{-1}$ has been used.

2.7 Numerical examples

Solutions of equations (2.6.4) are presented to illustrate the numerical procedures. In particular, the reference surface Ω is taken to be the annulus with the non-dimensional radii $r_i = 1$, $r_o = 2$. The undeformed configuration occupies the interval $r/r_i \in [1, 2]$. Solutions are obtained for various values of hub radius ρ_i. As shown later, approaches to the solutions vary depending on the states of deformations. In what follows, discussions on the solution procedure are given in detail.

For the given ρ_i, r_i and r_o, we first calculate the ratio ρ_i/r_i and compare it with the limits shown in (2.3.21). The relation of ρ_i/r_i to the limits determines the solution procedure.

If $\rho_i/r_i > (\rho_i/r_i)_1^{inc}$, it implies a certain amount of initial hub rotation to bring the membrane to the incipient state of wrinkling. To this end, we use equation (2.3.18) to determine the needed rotation τ^{inc}. The membrane remains fully tense for τ between zero and τ^{inc}. The solution in this region is determined explicitly by deformation (2.3.12) with constants (2.3.17). The torque M required to maintain the equilibrium is then calculated from formula (2.3.27). When the twist τ is larger than the incipient value τ^{inc}, the membrane becomes partly wrinkled and the equations (2.6.4) must be invoked to solve for β, λ_i and $\bar{\lambda}$. This task is undertaken by using a standard Newton-Raphson iterative algorithm for nonlinear algebraic equations. To implant this algorithm, we rearrange equations in (2.6.4) into the form $F_i(x_1, x_2, x_3) = 0$; $i=1, 2, 3$, where $x_1 = \beta$, $x_2 = \lambda_i$, and $x_3 = \bar{\lambda}$, and then calculate the elements $\partial F_i/\partial x_j$ of the Jacobian matrix (3×3) by

using the chain rule of differentiation with the help of equations (A.4), (A.12) and $(2.6.1)_{1,3}$.

To initialize the iterative procedure of determining β, λ_i and $\bar{\lambda}$ as a function of τ, we prescribe a small increment $\Delta\tau$ for τ from τ^{inc} such that $\tau = \tau^{inc} + \Delta\tau$. This will create wrinkling in a narrow band near the inner boundary r_i adjacent to the hub. Because the wrinkled band is very thin, by continuity, the variation of λ across the band is also small in magnitude. Thus the extreme values λ_i and $\bar{\lambda}$ of the stretch in the band are considered to be nearly equal. Guided by this, we start the iterations by setting both λ_i and $\bar{\lambda}$ equal to the value of $\lambda(r_i)$ obtained from $(2.3.13)_1$ with constants (2.3.24) evaluated at $\bar{r} = r_i$; $\bar{u}$ and $\bar{v}$ are replaced by those in $(2.3.16)_{1,2}$ evaluated at the incipient twist $\tau = \tau^{inc}$. Meanwhile, in view of (2.4.10), $\delta \approx 0$ due to the closeness of λ_i and $\bar{\lambda}$. Then $\psi \approx \theta$, thus, $\underline{i}(\psi) \to \underline{i}(\theta)$, $\underline{j}(\psi) \to \underline{j}(\theta)$, respectively. It is reasonable to substitute the tense solution (2.3.12) with $(2.3.16)_{1,2}$ into equation (2.1.18) to calculate the angle α at $r = r_i$ in the maximum state of τ^{inc} and use this value as the initial guess for β. Using these initial guesses to iterate system (2.6.4) until solutions of β, λ_i and $\bar{\lambda}$ are converged. We then prescribe an additional small increment in τ and repeat the above process, using the converged solution for the previous problem to supply the initial values β, λ_i and $\bar{\lambda}$ to initialize the present cycle of iterations. The procedure is terminated when τ is approximately equal to τ_{max} given by (2.4.36).

The foregoing solution scheme fails if $\rho_i/r_i = 1$. That is the case that the membrane is undeformed before a small increment twist $\Delta\tau$ is applied. According to the condition $(2.3.21)_1$ and theory for infinitesimal deformation, this is an incipient state such that a

small twist applied on the undeformed configuration results in the wrinkling of the entire membrane. It is found in our numerical computations that the Jacobian matrix associated with equations (2.6.4) becomes singular when ρ_i/r_i is near 1. Despite the singularity, using the finite-deformation theory, we have found that the algorithm is valid up to a certain value of ρ_i/r_i that is slightly less than unity. Moreover, in such cases wrinkling spreads rapidly with increasing τ when τ is small, however, it never covers the entire membrane.

If the prescribed value of $\rho_i/r_i < (\rho_i/r_i)_1^{inc}$, it implies that the membrane will be wrinkled before the twist is applied. The solution procedure is pursued in an alternative way. Under this circumstances, we divide the interval $(\rho_i/r_i, (\rho_i/r_i)_1^{inc})$ into small increments and compute a sequence of purely radial deformations in which τ, β, $\bar{\gamma}$, and $\bar{\delta}$ are all identically zero. As a result, the second equation of (2.6.4) becomes an identity and the system reduces to a set of two equations $(2.6.4)_{1,3}$ which are simplified considerably with $\tau = \beta = 0$. The associated Jacobian matrix is a 2×2 non-singular matrix. We need to calculate the value of λ_i in the state of incipient wrinkling. It is computed from $(2.3.13)_1$ in which the constants a_1, a_2, b_1 and b_2 are specified by boundary conditions (2.3.17) with $u_i/r_i = (\rho_i/r_i)_1^{inc}$, $v_i = 0$ obtained from $(2.3.16)_{1,2}$ by setting $\tau = 0$. To compute the first member of the sequence, we set ρ_i/r_i equal to a value that is slightly smaller than $(\rho_i/r_i)_1^{inc}$. The iterations are then started by setting both $\bar{\lambda}$ and λ_i equal to the value at the incipient state. Once the solution converges, the iteration cycles for subsequent members of the sequence are initiated by using the converged solution for the immediately preceding member. This process is terminated

when the prescribed value of ρ_i/r_i is reached. The solutions corresponding to the superposed twist are then pursued by prescribing the small increments in τ and the remainder of the solution proceeds in the manner described in the foregoing paragraphs.

Torque-twist relations are generated at each increment $\Delta\tau$ as twist τ progresses. When the membrane is fully tense, relation (2.3.28) is used to calculate the torque M. In the deformation that the membrane is partly wrinkled, it is convenient to use (2.4.25) to calculate the associated torque. The solutions of β and λ_i obtained can be used in equations $(2.6.1)_{2,3}$ to calculate a^* and b^* for fixed values of ρ_i/r_i and τ. These values in turn furnishes the non-dimensional torque M^* exerted by the hub on the membrane:

$$M^* = - M/(Gr_i^2) = - 2\pi a^* b^* (\rho_i/r_i)^2, \qquad (2.7.1)$$

where M is given by (2.4.25). The computed values of M^* are plotted as functions of τ in Figure 2.7.1 for values of ρ_i/r_o ranging from 0.10 to 0.45 at an interval of 0.5. Each curve terminates at the value of τ_{max}.

Increasing values of ρ_i/r_i reduces the prestretch in the membrane. As we discussed previously, for the larger value of ρ_i/r_i near 1 a relatively small twist produces tension lines that are inclined at large angles to the radial direction. This peculiar behaviour is analogous to the response of a sheared slab. If the slab is prestretched, the principal directions are seen to change orientations incrementally in respond to the incremental change in the amount of shear forces. However, these changes become significant as the prestretch decreases in intensity if the amount of shear is also small. Because of the large initial angles of the tension lines, i.e. large β in (2.4.21), it needs only a relatively small additional hub rotation to cause them to become tangential to the hub. As a result,

from (2.4.36), τ_{max} corresponding to the limiting state decreases with increasing ρ_i/r_i. It is shown by $(2.6.1)_{2,3}$, in the state of limiting twist, $b^* = -1$. The corresponding torque on the hub, from (2.4.25), is

$$M = 4\pi(1 - \lambda^{-2}_i)[(r_i/\rho_i)^2 \lambda^2_i - 1]^{1/2} \qquad (2.7.2)$$

In a state of less severe prestretch, $\lambda_i \to 1$, then the torque $M \to 0$ for the given ratio r_i/ρ_i near 1. Additionally, from (2.4.19), large values of ρ_i/r_i lead to large magnitudes of vector $\underline{u}$ ($|\underline{u}| = \rho_i$) which, in view of (2.4.24), represents the moment-arm for tension line $\underline{l}$. For large ρ_i/r_i, the membrane is less severely strained, but the large moment-arms of the tension lines results in a stiff torque-twist response. Conversely, when ρ_i/r_i is small, even though the membrane is more severely strained, this initial stiffening effect is overwhelmed by the relatively smaller moment-arms of the tension lines. The result is a softer torque-twist response. This is shown in Fig.2.7.1.

To avoid crowding, Fig. 2.7.1 does not show the torque-twist response for the case of extremely small prestretch ($\rho_i/r_i = 0.998$). In this case the membrane wrinkles almost immediately and the torsional stiffness is initially quite low. This stiffness increases very gradually due to the moment-arm effect. The result is a torque slightly smaller than that associated with $\rho_i/r_i = 0.9$ for small hub-rotation angles, but slightly larger for larger angles.

The growth of the wrinkled part of the membrane is shown in Fig. 2.7.2 for values of ρ_i/r_i from 0.2 to 0.9 at an interval of 0.1. Also plotted in Fig. 2.7.2 is the curve calculated for the extremely small prestretch $\rho_i/r_i = 0.998$. These results are presented by curves of ratio $\bar{r}/r_i$ against the hub rotation τ. Curves that intersect the horizontal axis

correspond to the states that the membrane remains tense up to τ^{inc}. This is the distance from the origin to the intersection point. Variations of $\bar{r}/r_i$ corresponding to the state of partial wrinkling without twist ($\tau = 0$) are illustrated by curves that intersect the vertical axis. It can be observed that curves for $\rho_i/r_i = 0.6, 0.998$ nearly correspond to the incipient states indicated by $(2.3.21)_{2.1}$ respectively. On close inspection, for small ρ_i/r_i, it is evident that $\bar{r}$ varies slowly over a large range of hub rotation, whereas this variation becomes acute over a relatively small rotation angle for large ρ_i/r_i. More particularly, in the case of slight prestretch ($\rho_i/r_i = 0.998$), the membrane is wrinkled almost immediately for an infinitesimally small amount of rotation. The subsequent wrinkling progresses rapidly as the twist τ increases when τ is still small. Wrinkling dwindles substantially as τ_{max} is approached. In contrast, for $\rho_i/r_i = 0.6$, the rate of growth of $\bar{r}$ is seen to be relatively slow. In all examples that we have considered, $\bar{r}$ is the monotone increasing function of rotation τ as τ increases. After overcoming the incipient state, wrinkling proceeds to consume an ever increasing portion of the membrane. However, it is predicted by the tension field theory that the membrane will never be wrinkled entirely. We make another observation that the difference between τ_{max} is quite substantial, the difference between the largest and smallest values is about 40°.

Fig. 2.7.3 and Fig. 2.7.4 show the distributions of the principal stretches λ and μ in the membrane at the limiting condition $\tau = \tau_{max}$. Numerical values are calculated along the reference radius r for each value of ρ_i/r_i from 0.2 to 0.9 at an interval of 0.1. From Fig. 2.7.3, it is evidently shown that the largest principal stretch λ in the limiting state occurs at the hub ($r/r_i = 1$) while ρ_i/r_i being the smallest. This is the largest stretch

among all of the examples we have considered. Its value ($\lambda = 2.55$) lies within the range in which the uniaxial constitutive relation $f(\lambda)$ defined in $(2.2.3)_1$ approximates the behaviours of rubber with reasonable accuracy. Finally, the variation of the invariant J ($=\lambda\mu$) along the reference radius r in the limiting states is shown in Fig. 2.7.5. These results of J confirms our assumption that wrinkling ($J < 1$) occurs in an annular region adjacent to the hub.

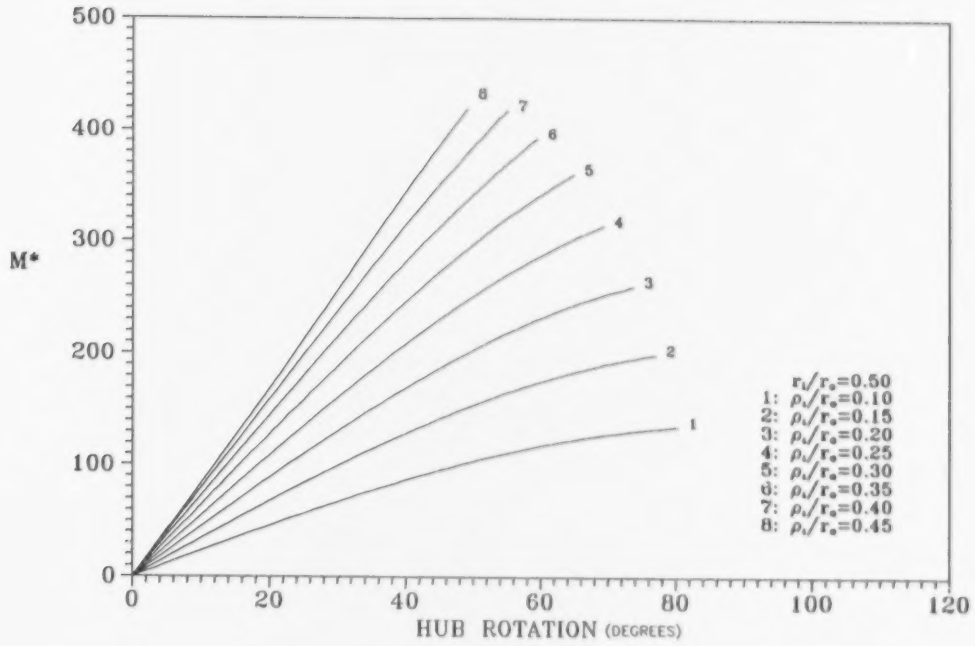

Fig. 2.7.1 Equilibrium torque-twist response

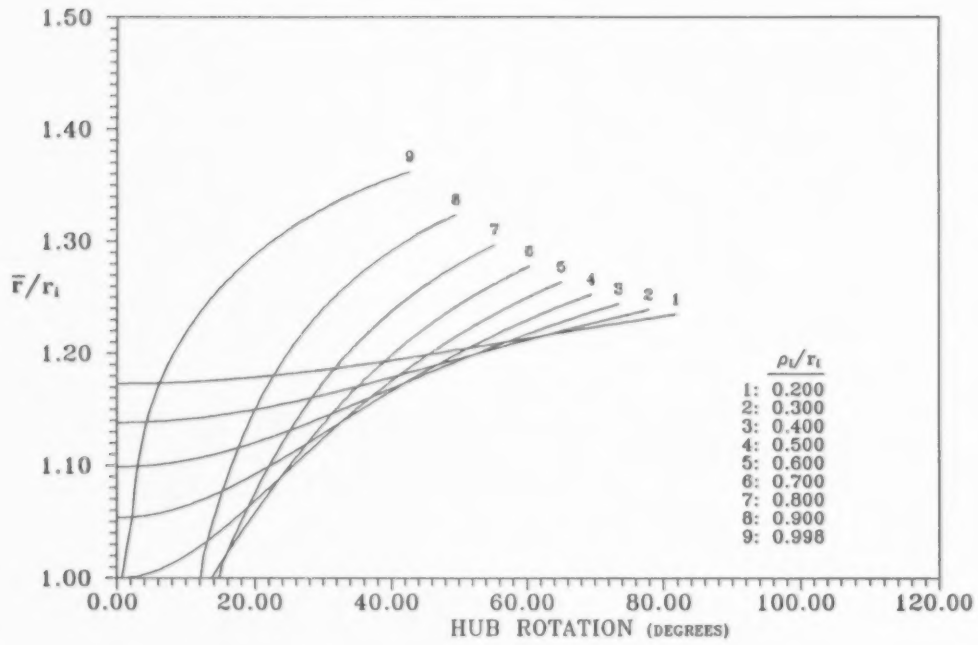

Fig. 2.7.2 Evolution of the boundary $\bar{r}/r_i$ separating tense and wrinkled parts of the membrane

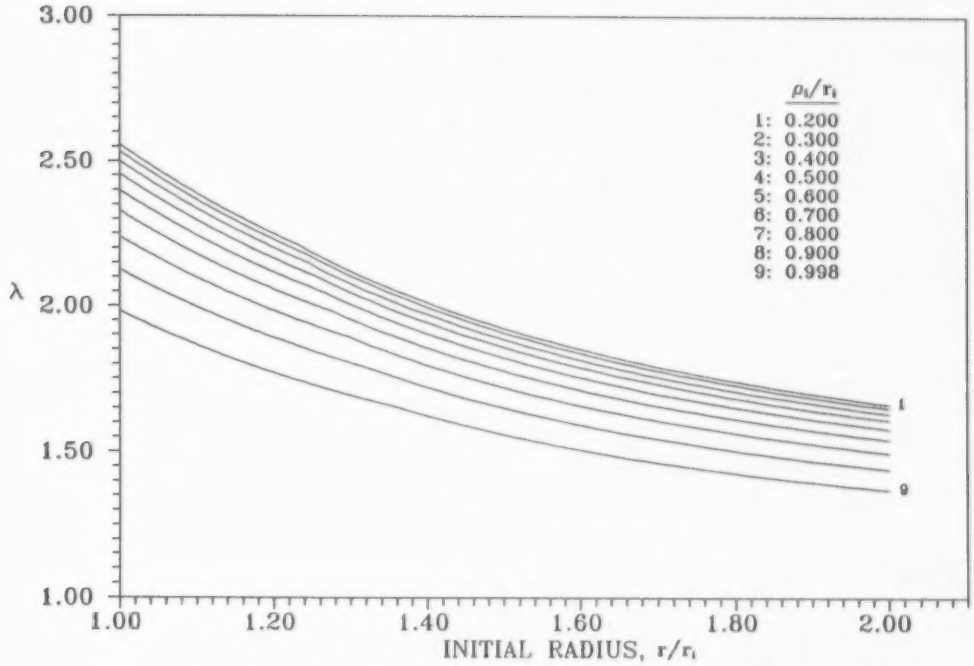

Fig.2.7.3 Variation of λ with initial radius at $\tau = \tau_{max}$

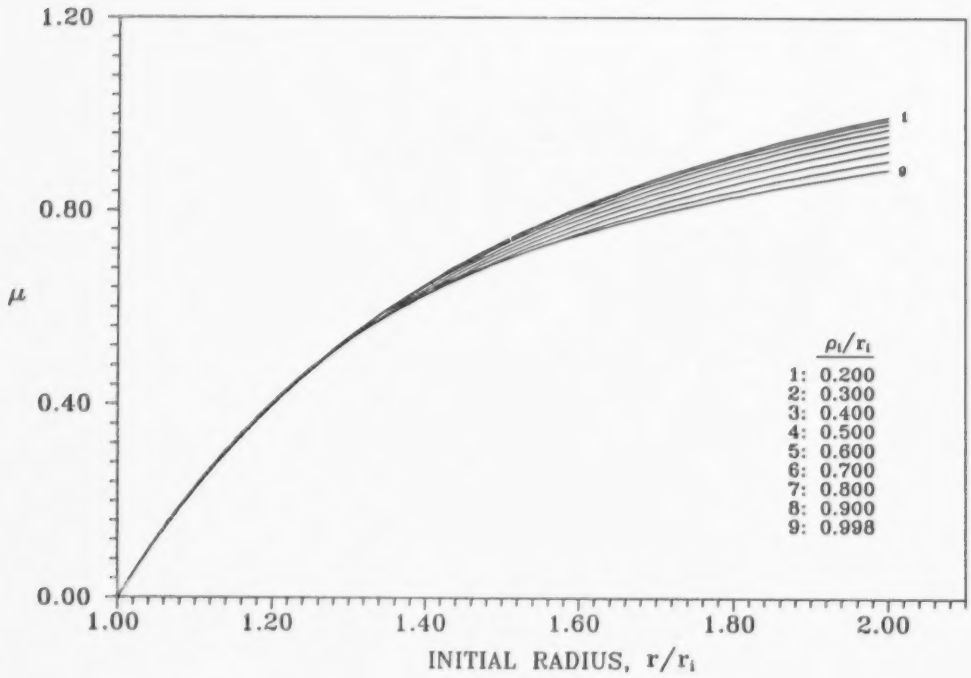

Fig. 2.7.4 Variation of μ with initial radius at $\tau = \tau_{max}$

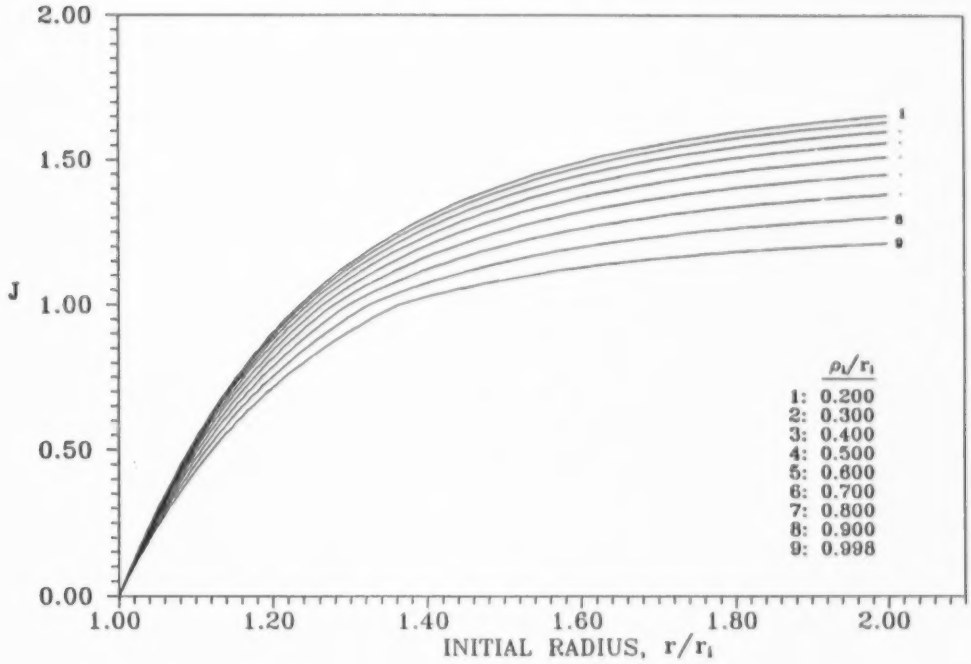

Fig.2.7.5 Variation of J with initial radius at $\tau = \tau_{max}$. The membrane is wrinkled where $J < 1$ and tense where $J > 1$

CHAPTER 3 AXISYMMETRIC DEFORMATINS OF A SURFACE OF REVOLUTION

In the present work we consider the large deformations of elastic membrane surfaces of revolution subject to axisymmetric loadings. External forces applied to the membrane are composed of spatially uniform internal pressure and concentrated force (point-load). The membrane is assumed to be made of elastic material which behaves nonlinearly. Using tension field theory, we consider the wrinkling effects which arise in the deformations. To do that, in section 3.1, we adapt Ogden's 3-term strain energy function associated with three-dimensional elasticity into a form which is suitable in two-dimensional membrane theory. Based on this, the relaxed strain energy function is obtained. Section 3.2 is devoted to the derivation of kinematic relations for the axisymmetric deformations of surfaces of revolution. Also in this section, loading potential functions of conservative forces applied on the membrane are given and a general expression for the total potential energy is established. Using the minimum potential energy principle, equilibrium equations that describe axisymmetric deformations of surfaces of revolution can be obtained for each individual problem involving different geometry and external force. In section 3.3, we have derived the equations of equilibrium for a hemispherical membrane under the action of an internal pressure as well as a concentrated force at the pole. The equations are, then, written in a suitable form for numerical integrations. Using the obtained equations, in section 3.4, we consider the inflation of a hemispherical membrane subjected to a uniform internal pressure alone. The total potential energy is used to resolve an issue of non-uniqueness. In section 3.5, wrinkling analysis has been conducted for the problem of a hemispherical

membrane under the action of point-loads at its pole. In this section, we are able to demonstrate that, for the reference configuration being a surface of revolution, wrinkling analysis may be further simplified by the special features of the relaxed strain energy function. As a result, the system of governing equations is reduced to a quadrature in the wrinkled region. It is shown later that point-load problems always involve singularities in the governing equations in their original forms, the meridional stretch becomes infinity near the point of application of the point-load. In this work, we are able to show that using the relaxed strain energy function of Ogden's type [1.22], the singularity is removed once the governing differential equations are transformed into an integral equation. To illustrate the method, we applied the integral equation to the examples of point-loads for the pressurized and non-pressurized hemispherical membranes. In section 3.6, we consider a truncated hemispherical membrane pulled apart by edge loads. In this example, the deformed surface is wrinkled in an open interior region whereas regions adjacent to the rigid boundaries are tense. In section 3.7, following Steigmann's formula [3.1] and using the quadrature derived in section 3.5, we consider a special case in which an initially closed circular membrane is pulled apart at its apices onto a straight line along the axis of symmetry. The deformed membrane is thus entirely wrinkled. The force-deflection relation is given in the form of an integral. Numerical examples for various values of force are tabulated. Section 3.8 considers deformations of a cylindrical tube which is stretched at one end into a flat circular sheet, followed by contracting the other end to the point of symmetry in the same plane. The solutions to the governing equations are solved by a semi-inverse method. Finally, in section 3.9, we consider a collapsed cylindrical membrane torus which is

axisymmetrically inflated by the application of the internal pressure. In this case, we again have a non-uniqueness relation between the applied pressure and the deformation. This is resolved by comparing the potential energy associated with each of the deformation. However, particular interest is focused on the solution procedures and the wrinkling effects.

3.1 Ogden's three-term strain energy function

To complete the governing equations of equilibrium, constitutive relations are needed. We assume that the membrane is homogeneous, isotropic and made of so-called Ogden's material whose strain energy function, the value of which corresponds to the strain energy stored in the deformed membrane and measured per unit reference area, can be expressed as a series of powers in principal stretches λ and μ. As shown later, the strain energy function expressed in this form has particular advantages in solving point-load problems which involves the pole-type singularity in the differential equations as it is evaluated near the pole. In the moderate range of strain ($\lambda < 7$), Ogden's energy function is in good agreement with experimental results for incompressible rubber-like solids. A detailed discussion about this material can be found in the book by Ogden [1.20].

It is shown by Ogden [1.20] that experimental data for tension of solid rubberlike materials can be described by the strain energy function

$$U = G^{*} \sum_{r=1}^{3} g_r(\lambda_1^{\alpha_r} + \lambda_2^{\alpha_r} + \lambda_3^{\alpha_r} - 3)/\alpha_r, \qquad (3.1.1)$$

which is measured per unit volume of reference body. λ_i (i=1,2,3) are the principal stretches, G^* is the bulk shear modulus for infinitesimal strains from the natural unstrained state. Material constants are given by

$$\alpha_1 = 1.3,\ \alpha_2 = 5.0,\ \alpha_3 = -2.0\ ;$$

$$g_1 = 1.491,\ g_2 = 0.003,\ g_3 = -0.0237 \qquad (3.1.2)$$

Within the context of Naghdi's direct surface theory [1.15], a membrane is modelled as a two-dimensional elastic continuum endowed with strain energy density per unit area of the reference surface. In the direct theory for an isotropic membrane, the strain energy function, as shown in equation (1.2.36), can be expressed as a symmetric function of two principal stretches which are the length ratios of material elements aligned with the principal directions of strain in the tangent planes of the deformed and reference surfaces. This implies that the bulk properties of the material do not enter into the constitutive relations explicitly. Accordingly, the strain energys for both compressive and incompressible materials can be described by a single function without kinematic constraints. For Ogden's material characterized by the strain energy function of three dimensional elasticity as shown in equation (3.1.1), the corresponding strain energy function for membrane theory can be derived by imposing the conditions $\lambda_1 = \lambda$, $\lambda_2 = \mu$ and $\lambda_3 = 1/\lambda\mu$ in the function $U(\lambda_1,\lambda_2,\lambda_3)$. Thus

$$w(\lambda,\ \mu) = G \sum_{p=1}^{3} g_p\left[\lambda^{\alpha_p} + \mu^{\alpha_p} + (\lambda\mu)^{-\alpha_p} - 3\right]/\alpha_p, \qquad (3.1.3)$$

where $G = G^*h$ and h is the initial thickness of the membrane.

Components of Piola stress tensor as shown in (1.3.37) can be computed from the following derivatives,

$$w_\lambda(\lambda,\ \mu) = G \sum_{p=1}^{3} g_p\left[\lambda^{(\alpha_p-1)} - \mu^{-\alpha_p}\lambda^{-(\alpha_p+1)}\right] \tag{3.1.4}$$

$$w_\mu(\lambda,\ \mu) = G \sum_{p=1}^{3} g_p\left[\mu^{(\alpha_p-1)} - \lambda^{-\alpha_p}\mu^{-(\alpha_p+1)}\right] \tag{3.1.5}$$

Second derivatives will be needed later. By differentiating (3.1.4, 5) with respect to λ and μ, respectively, we have

$$w_{\lambda\lambda}(\lambda,\ \mu) = G \sum_{p=1}^{3} g_p\left[(\alpha_p-1)\lambda^{(\alpha_p-2)} + (\alpha_p+1)\mu^{-\alpha_p}\lambda^{-(\alpha_p+2)}\right] \tag{3.1.6}$$

$$w_{\lambda\mu}(\lambda,\ \mu) = G \sum_{p=1}^{3} g_p\alpha_p(\lambda\mu)^{-(\alpha_p+1)} \tag{3.1.7}$$

The relaxed strain energy function of the elastic membrane is defined as the strain energy evaluated when the membrane is in a state of uniaxial tension. Let λ be the principal stretch corresponding to the tensile uniaxial stress, the associated transverse stretch μ at this state is defined as the natural width $\upsilon(\lambda)$. Thus the relaxed strain energy function is defined as $w^R(\cdot) = w(\lambda,\upsilon(\lambda))$. The value of the natural width $\upsilon(\lambda)$ is a material property. To determine its expression for a given material, let's consider the simple tension of a unit cube consisting of isotropic, incompressible elastic material. If λ is the stretch in the direction of tension, the transverse stretches λ_2 ($= \mu$) and λ_3 ($= 1/\lambda\mu$) must be equal. Thus, by setting $\lambda_2 = \lambda_3 = \mu$, we obtain the natural width as

$$\upsilon(\lambda) = \lambda^{-1/2}. \tag{3.1.8}$$

Evaluating (3.1.3) at the natural width of simple tension, i.e. $\mu = \upsilon(\lambda)$, the relaxed strain energy is then obtained as

$$w^R(\lambda) = G \sum_{r=1}^{3} g_r(\lambda^{\alpha_r} + 2\lambda^{-\alpha_r/2} - 3)/\alpha_r, \tag{3.1.9}$$

Since the relaxed strain energy $w^R(\cdot)$ is inherently independent of the principal stretch μ corresponding to which the principal stress is identically zero, their derivatives possess following properties

$$w^R_{\lambda} = G \sum_{r=1}^{3} g_r\left(\lambda^{\alpha_r - 1} - \lambda^{-(\alpha_r+2)/2}\right)$$
$$w^R_{\mu} = 0 \qquad (3.1.10)$$

and

$$w^R_{\lambda\lambda} = G \sum_{p=1}^{3} g_p \left[(\alpha_p - 1)\lambda^{\alpha_p - 2} + (\frac{1}{2}\ \alpha_p + 1)\lambda^{-(\frac{\alpha_p}{2}+2)}\right] \qquad (3.1.11)$$
$$w^R_{\lambda\mu} = 0$$

Behaviours of $w^R(\lambda)$, $w^R_{\lambda}(\lambda)$ and $w^R_{\lambda\lambda}(\lambda)$ are plotted against λ in Fig.3.1.1.

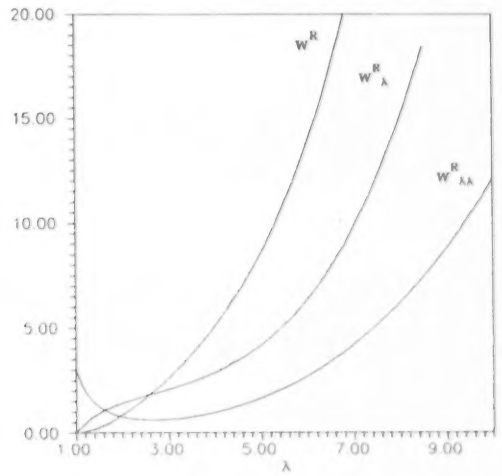

Fig.3.1.1 Behaviours of the Relaxed energy of Ogden's type.

Equation $(3.1.10)_2$ implies that the relaxed strain energy yields a uniaxial state of stress in the tension fields. It is easily verified that the relaxed energy defined by (3.1.9)-

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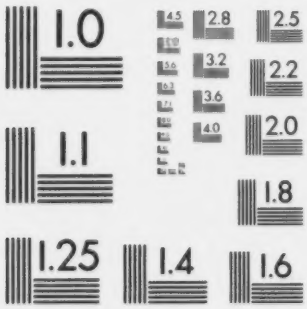

PRECISION[SM] RESOLUTION TARGETS

(3.1.11) automatically satisfies necessary conditions (1.3.10) and (1.3.11) in those parts of the (λ, μ)-plane where $\lambda > 1$, $0 \leq \mu \leq \upsilon(\lambda)$. Due to the complicated form of (3.1.3) it is not feasible to check these inequalities analytically in the region corresponding to the first branch of (1.3.26). Instead, we calculate these inequalities numerically and verify that they are satisfied at all mesh points in our numerical solutions.

3.2 Kinematics and potential energy

In this section we review some essential kinematic relations associated with the axisymmetric deformations of surfaces of revolution [3.1]. For a surface of revolution, its geometry can be described by a cylindrical polar coordinate system. In addition, on the surface, we can also introduce a pair of orthogonal coordinates (θ,s) to label the material point, where θ is the azimuthal angle and s measures the arc-length along the meridian. Thus, we take (θ,s) to be the Larangian coordinates which are used as the independent variables in the formulation of governing equations. To establish kinematic relations, we take the reference configuration to be a surface which is generated by a plane curve C_0 through a complete revolution about axis $\underline{k}$. We employ a cylindrical polar coordinate system where r is the radial coordinates, θ is the azimuthal angle, z is the axial coordinate defined in the usual notation. Fig.3.2.1. shows the cylindrical portion of an axial cross section of the meridional curve cut off by two circles of latitude.

For the reference that is generated by rotating a plane curve C_o, it can be described parametrically by the position vector

$$\underline{x}(\theta, s) = r(s)\underline{i}(\theta) + z(s)\underline{k} \; ; \; r \geq 0, \tag{3.2.1}$$

where $\theta \in [0,2\pi]$, s measures arclength along the meridian, $\underline{k}$ is a fixed unit vector along the axis of revolution and $\underline{i}(\theta)$ is a unit vector directed radially outward from the axis.

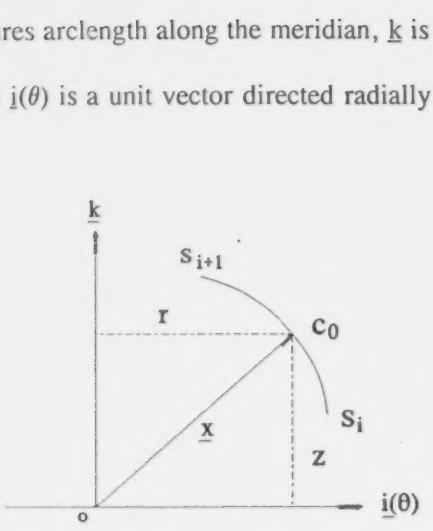

Fig. 3.2.1 Cylindrical portion of the reference surface

In an axisymmetric deformation of a surface of revolution, the particle $\underline{x}$ with coordinates (θ,s) is displaced to the new place defined by the position vector

$$\underline{y}(\theta, s) = \rho(s)\underline{i}(\theta) + \zeta(s)\underline{k} \; ; \; \rho \geq 0, \tag{3.2.2}$$

where functions $\rho(s)$ and $\xi(s)$ are to be determined.

In view of the reference surface given by equation (3.2.1), the deformation described in (3.2.2) shows that the reference meridional curves are deformed in the θ-plane. The circles of latitude (s = const) are radially expanded as well as displaced vertically. The pointwise orthogonality of the meridians to the circles of latitude is thus preserved after the deformation. Unless explicitly stated otherwise, functions $\rho(s)$ and $\xi(s)$ are assumed to be continuously differentiable with respect to the arclength s. This condition thus ensures the smoothness of the deformed meridian curves. Fig.3.2.2 sketches the deformed meridional curves.

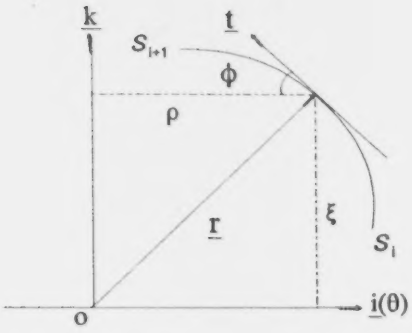

Fig.3.2.2 Cylindrical portion of the defromed surface

The deformation gradient $\underline{F}$ can be obtained by first differentiating (3.2.2) to obtain

$$d\underline{y}(\theta, s) = \underline{t}ds + \rho\underline{j}(\theta)d\theta, \tag{3.2.3}$$

where $\underline{t} = [\rho'(s)\underline{i}(\theta) + \zeta'(s)\underline{k}]$ is the tangent to the deformed meridian as shown in Fig.3.2.2. ($\underline{t}$ is not a unit vector, in general). $\underline{j} = \underline{i}'(\theta) = \underline{k} \times \underline{i}(\theta)$ is the unit tangent to a circle of latitude (s = const). Expressions for the increments ds and $d\theta$ can be obtained by an analogous expression based on (3.2.1). We have

$$d\underline{x} = \underline{T}\, ds + r\underline{j}(\theta)d\theta. \tag{3.2.4}$$

where $\underline{T} = [r'(s)\underline{i}(\theta) + z'(s)\underline{k}]$ is the unit tangent to the reference meridian C_o. Scalar multiplying (3.2.4) by $\underline{T}$ and $\underline{j}(\theta)$ yields relations

$$ds = [r'(s)\underline{i}(\theta) + z'(s)\underline{k}] \cdot d\underline{x} \quad \text{and} \quad d\theta = r^{-1}\underline{j}(\theta) \cdot d\underline{x}, \tag{3.2.5}$$

respectively. Where $\underline{T} \cdot \underline{j}(\theta) = 0$ has been used. The centre dot denotes the scalar product of two vectors.

Substituting (3.2.5) into (3.2.3) and comparing the result with the defining equation (1.2.15) furnishes $\underline{F}$ in the form,

$$\underline{F} = [\rho'(s)\underline{i}(\theta) + \zeta'(s)\underline{k}] \otimes [r'(s)\underline{i}(\theta) + z'(s)\underline{k}] + (\rho/r)\underline{j}(\theta) \otimes \underline{j}(\theta) \tag{3.2.6}$$

The Cauchy-Green strain tensor is defined in (1.2.20), in view of (3.2.6), we obtain

$$\underline{C} = [\rho'(s)^2 + \zeta'(s)^2]\, \underline{T}\otimes\underline{T} + (\rho/r)^2 \underline{j}(\theta)\otimes\underline{j}(\theta), \tag{3.2.7}$$

and its fundamental scalar invariants are

$$\left.\begin{aligned} J &= (\det\underline{C})^{1/2} = [\rho'(s)^2 + \zeta'(s)^2]^{1/2} \cdot (\rho/r); \\ I &= (\mathrm{tr}\underline{C} + 2J)^{1/2} = [\rho'(s)^2 + \zeta'(s)^2]^{1/2} + (\rho/r) \end{aligned}\right\} \tag{3.2.8}$$

and

$$I = \lambda + \mu \quad \text{and} \quad J = \lambda\mu \tag{3.2.9}$$

where λ^2 and μ^2 are the principal stretches associated with $\underline{C}$. We assume that $\lambda > 0$ and $\mu > 0$. Solving (3.2.9) for λ (>0) and μ (>0) and using equations (3.2.8) for I and J, the meridional and circumferential stretches are related to the surface geometry through relations

$$\lambda = [\rho'(s)^2 + \zeta'(s)^2]^{1/2} \quad \text{and} \quad \mu = \rho/r \tag{3.2.10}$$

λ and μ measure the stretches of the meridians and circles of latitude during the deformation. Relations (3.2.10) show that the principal variables λ and μ are independent of θ and, therefore, may be regarded as functions of the arclength s.

Equation (1.2.22) expresses the deformation gradient $\underline{F}$ in terms of the principal stretches λ, μ and the associated principal vectors $\{\underline{L},\underline{M}\}$, $\{\underline{l},\underline{m}\}$. Based on equation (3.2.6) and (3.2.10), we are able to obtain the following relations,

$$\left.\begin{aligned} \underline{L} &= \underline{T} = r'(s)\underline{i}(\theta) + z'(s)\underline{k}, \qquad \underline{M} = -\,\underline{j}(\theta) \\ \lambda\underline{\ell} &= \underline{t} = \rho'(s)\underline{i}(\theta) + \zeta'(s)\underline{k}, \qquad \mu\underline{m} = -\,(\rho/r)\underline{j}(\theta). \end{aligned}\right\} \tag{3.2.11}$$

Here we have chosen $\underline{L}$ and $\underline{M}$ such that $\underline{L}$ is the unit tangent to the reference meridian and $\underline{N} = \underline{L} \times \underline{M}$ is the exterior unit normal vector. By axisymmetry of $(3.2.11)_{3,4}$, $\{\underline{l}, \underline{m}\}$ are tangential to the meridians and circles of latitude on the deformed surface.

As demonstrated by Fulton [3.2] and Yang [3.3], an introduction of a geometric variable enables us to describe the equilibrium state by a system of first order differential equations. To show that, let $\phi(s)$ be the angle formed by the unit tangent $\underline{l}$ to the deformed meridian with the inward radial axis $-\underline{i}(\theta)$, upon introducing ϕ, unit vector $\underline{l}$ can be resolved as

$$\underline{\ell} = -\cos\phi\ \underline{i}(\theta) + \sin\phi\ \underline{k}. \tag{3.2.12}$$

where ϕ is shown in Fig.3.2.2.

Substituting (3.2.12) into $(3.2.11)_3$ and grouping the coefficients for each base vectors $\underline{i}(\theta)$, $\underline{k}$, we have

$$\rho'(s) = -\lambda\cos\phi,\quad \zeta'(s) = \lambda\sin\phi. \tag{3.2.13}$$

According to (3.2.13), λ, μ and ϕ are not independent, but must satisfy a certain compatibility equation. This condition is supplied by equation $(3.2.10)_2$ in the differential form

$$r\mu' = -\lambda\cos\phi - \mu r'. \tag{3.2.14}$$

Equation (3.2.14) will be used as one of the governing equations in the problems to be considered later.

Equations of equilibrium can be obtained by applying minimum energy principle on the total potential energy of the deformation if the membrane is subjected to a system of conservative forces. Two particular types of external forces are considered. They are, namely, the internal pressures and the point-loads. We further restrict that the intensities of the forces are constants independent of the deformations they have caused.

For an internal pressure of constant intensity, Steigmann [1.23] proved the existence of the potnetial energy function which can be calculated by an integral expressed in

equation (1.2.38). We assume that the internal pressure is spatially uniform with the intensity p^d. The assumption that the intensity p^d remains constant during the process of deformations approximates to the situation in which the gas in the interior of the membrane is connected to a very large reservoir of the gas. In order to calculate the potential energy associated with the internal pressure, we need to know the volume enclosed by the deformed membrane. For this type of conservative force, Steigmann [3.2] had derived the potential energy function and it is presented in (1.2.38). In particular, to construct the potential energy for axisymmetric deformations under consideration, let the principal vectors $\underline{\ell}$ and $\underline{m}$ defined by $(3.2.11)_{3,4}$ be oriented so that $\underline{n} = \underline{\ell} \times \underline{m}$ is the outward unit normal vector to the deformed surface. Substituting equations (3.2.2) and $(3.2.11)_{3,4}$ for $\underline{y}$, $\underline{l}$ and $\underline{m}$ in (1.2.38), we obtain the integrand in the form

$$J\underline{n} \cdot \underline{y} = (\lambda\underline{\ell}) \times (\mu\underline{m}) \cdot \underline{y} = r^{-1}\rho(\rho\zeta' - \zeta\rho'). \tag{3.2.15}$$

Based on (3.2.15), the loading potential function $P_p[\underline{y}]$ for the cylindrical portion [0, L] of the reference membrane above the equator is obtained. The elemental area $dA = r\,ds$. The term $\rho^2\xi$ goes out on integrating over the arclength, we arrive at the equation

$$P[\underline{y}]/2\pi = \frac{2p}{3}(\rho^2\zeta)\Big|_0^L + p\int_0^L [\frac{1}{2}\rho^2\zeta']ds. \tag{3.2.16}$$

Next, suppose that an axial dead point-load $\underline{F}^d_c = F^d_c\,\underline{k}$ is axisymmetrically applied on the surface, where F^d_c is the magnitude of the force. In the problems that the reference surface is either a hemisphere or a closed circular plane, $\underline{F}^d_c$ is applied vertically at their apices. In the problems that the reference surface is a truncated hemisphere, $\underline{F}^d_c$ is applied vertically at the centre of the rigid disc attached at the truncation. For all the

problems we mentioned, the loading potential function associated with the point-load $\underline{F}^d_c$ can be presented in the form $P^c = \underline{F}^d_c \cdot \zeta(L)$, where L is the arclength at which $\underline{F}^d_c$ is applied. Since F^d_c is a constant, the loading potential P^c can be converted into an integral.

So as to be able to obtain the equilibrium equations from the total potential energy of the deformation, it is now assumed that the elastic membrane is homogeneous, isotropic and has a strain energy function of the form of $w(\lambda,\mu)$. Therefore, the total potential energy can be written as one integral. Adding the total strain energy of the deformed membrane and the two loading potentials, we write the functional of the deformation

$$E/2\pi = \int_0^L H(\rho,\ \rho',\ \xi,\ \xi';\ s)\,ds\ -\ \frac{2p^d}{3}(\rho^2\xi)\Big|_0^L\ -F\xi(0) \qquad (3.2.17)$$

where

$$\left.\begin{aligned} H &= r(s)w(\lambda,\mu) - (\tfrac{1}{2}p^d\rho^2 + F))\xi' \\ F &= F^d_c/2\pi. \end{aligned}\right\} \qquad (3.2.18)$$

For the deformation in which the equator of the membrane remains fixed, $\xi(0)$ becomes zero and the last term in (3.2.17) vanishes identically.

Following relations will be useful in expressing the Euler-Lagrange equations in terms of the principal stretches. In view of equations (3.2.10), the strain energy $w(\cdot)$ may be expressed as the function of ρ',ρ and ξ':

$$w(\lambda,\mu) = w(\rho',\rho,\xi'). \qquad (3.2.19)$$

For the sake of notational convenience, we use the same functional notation $w(\cdot)$ for two different functions representing the strain energy function. They are distinguished by the explicit appearance of arguments.

Again, using relations (3.2.10), the chain rule of differentiation is used to derive equations like

$$w_{\rho'} = \rho'\lambda^{-1}w_\lambda, \quad w_{\zeta'} = \zeta'\lambda^{-1}w_\lambda; \qquad w_\rho = r^{-1}w_\mu. \tag{3.2.20}$$

which are used in conjunction with the compatibility equations (3.2.13) to obtain the following derivatives:

$$w_{\rho'} = -\, w_\lambda \cos\phi, \quad w_{\zeta'} = w_\lambda \sin\phi. \tag{3.2.21}$$

The equations of equilibrium (Euler-Lagrangean equations) can now be obtained directly from (3.2.17), provided that the boundary conditions are appropriately specified. The governing equations can be reduced to a system of ordinary differential equations which may be integrated by numerical methods. It is evident that approaches to the solutions of these differential equations will be influenced markedly by the reference geometry, i.e. r(s), the loading parameters p^d, F, and the nature of strain energy function $w(\cdot)$. In subsequent sections, we will consider in detail the deformations for various surface geometries and loading parameters.

3.3 A hemispherical membrane

In section 3.4, 5, 6, we will consider axisymmetrical deformations of a hemispherical membrane. The axial cross-section of the reference surface is shown in Fig.3.3.1. R is the radius. The arclength s is measured along the meridian from the equator z = 0. t is the elevation angle measured counterclockwisely from $\underline{i}$-axis. Considering the reference surface as shown in Fig.3.3.1, the meridian of the surface is described by the position vector (3.2.1). Its components and the range of parameter is specified by relations

$$r(s) = R \cos (s/R), \quad z(s) = R \sin (s/R) \; ; \; s \in [0, \pi R/2], \qquad (3.3.1)$$

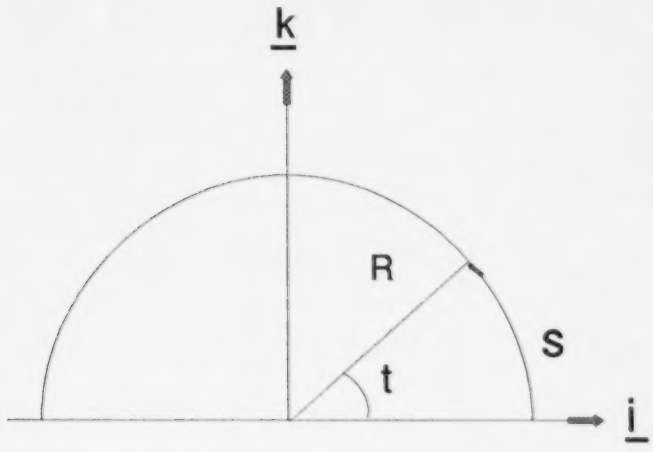

Fig.3.3.1 A hemispherical surface

In the following work, we confine ourselves to the class of axisymmetrical deformations described by (3.2.2). Whereby, kinematic relations derived in section 3.2 in connection with surfaces of revolution are readily applicable here. In the sequel, these relations will be frequently referred to without explicit explanation.

To obtain the Euler-Lagrangean equations for the potential energy (3.2.17), appropriate boundary conditions must be specified. We consider the deformations in which the equator is fixed, its radial and axial displacements are zero. In addition, by axisymmetry, the radial displacement of the pole must be zero. The corresponding boundary conditions are summarized as

$$\rho(0) = R, \quad \zeta(0) = 0 \quad \text{at equator}$$

$$\rho(R\pi/2) = 0 \quad \text{at the pole.} \qquad (3.3.2)$$

Suppose that the membrane is subjected to the internal pressure which is applied normally to the inside of the surface so as to ensure that the deformation of the

membrane is axisymmetric. An vertically-directed axial point-load is also applied at the pole. It should be mentioned that the application of point-loads does not necessitate the presence of internal pressure. In fact, we will consider the deformations caused by point-loads alone. Using p^d and F^d_c to denote the internal pressure and the point-load at the pole respectively, and taking account of the boundary conditions (3.3.2), the total potential energy of the deformation can be particularized from (3.2.17) and it takes the form

$$E/2\pi = \int_0^{\frac{\pi R}{2}} \left[r(s)w(\lambda,\mu) - \frac{1}{2}p^d\rho^2\xi' \right] ds - F^d{}_c\xi(\frac{\pi R}{2}) \tag{3.3.3}$$

Following the familiar procedure of variational theory, we obtain the Euler-Lagrange equations

$$\left.\begin{aligned} (r\cos\phi\, w_\lambda)' &= p^d \rho\lambda \sin\phi - w_\mu \\ r\sin\phi\, w_\lambda &= \tfrac{1}{2}\, p^d\rho^2 + F^d_c. \end{aligned}\right\} \tag{3.3.4}$$

in which relation (3.2.10), (3.2.13) and (3.2.21) have been used. ϕ is the geometrical variable defined by equation (3.2.12). The prime denotes the differentiation with respect to the arclength s. Equilibrium equations (3.3.4) are obtained without reference to the geometry of the reference configuration, so it should be valid for axisymmetric deformations of any surfaces of revolution. In the latter part of this section, we particularize these equations to the reference configuration as a hemispherical surface.

Equations (3.3.4) alone are not sufficient to solve for three unknowns λ, μ and ϕ. To establish a closed-system of solving equations, an additional equation is needed. To supplement, either the compatibility equation $(3.2.13)_1$ or (3.2.14) can be included. In the end, the set of equations (3.3.4) together with the supplementary equation suffice to

determine the three unknowns λ,μ and ϕ and constitutes a system of differential-algebraic equations.

We generalize the formulation by non-dimensionalizing the geometrical and loading quantities with respect to the reference radius R and the material constant G. The non-dimensional quantities are defined by

$$\begin{aligned} &t = s/R, \quad \hat{r} = r/R, \quad \hat{\rho} = \rho/R, \quad \hat{\xi} = \xi/R; \\ &\hat{F}^d_c = F^d_c/GR, \quad \hat{p}^d = p^d R/G, \quad \hat{w} = w/G, \end{aligned} \tag{3.3.5}$$

where t is the dimensionless arclength, or the elevation angle as shown in Fig.3.3.1. G is the material constant associated with the strain energy defined by (3.1.3). Henceforth, we will take t to be the independent variable. For notational convenience, non-dimensional quantities are used with the hats being dropped.

As a result of non-dimensionalization, the dimensionless version of compatibility equation (3.2.13) is

$$\dot{\rho}(t) = -\lambda \cos\phi \; ; \; \dot{\xi}(t) = \lambda \sin\phi \tag{3.3.6}$$

where the superposed dot denotes the differentiation with respect to the elevation angle t.

Accordingly, the compatibility equation (3.2.14) becomes

$$\dot{\mu} = -(\lambda \cos\phi + \mu\dot{r})/r, \tag{3.3.7}$$

Moreover, the non-dimensional boundary conditions (3.4.2) are

$$\rho(0) = 1, \quad \xi(0) = 0 \quad \text{and} \quad \rho(\pi/2) = 0. \tag{3.3.8}$$

Governing equations expressed in the analogous forms as those in (3.3.4) were derived in a number of references [3.2,3.6]. However, we would like to present them in a slightly different form which has proved to be extremely convenient for carrying out the

wrinkling analysis. This is accomplished by substituting rw_λ obtained from $(3.3.4)_2$ into the bracket on the left side of $(3.3.4)_1$ and carrying out the differentiation to get

$$\dot{\phi} = \{p^d\rho\lambda - w_\mu \sin\phi) / (rw_\lambda), \tag{3.3.9}$$

We then differentiate equation $(3.3.4)_2$, and replace $\dot{\phi}$ by the expression (3.3.9) to obtain two equations

$$\left.\begin{aligned} (rw_\lambda)^{\cdot} &= - w_\mu \cos\phi \\ (rw_\lambda)\sin\phi &= \tfrac{1}{2}\, p^d\rho^2 + F^d_c, \end{aligned}\right\} \tag{3.3.10}$$

which are functionally equivalent to the set of equations in (3.3.4). It is evident that there are no explicit presence of ϕ, p^d on either the left-hand or right sides of the equation $(3.3.10)_1$.

An alternate, and equally useful form of $(3.3.10)_1$ for the effective implementation of numerical scheme is to express $\dot{\lambda}$ and $\dot{\xi}$ in terms of λ and μ. It is known that, for homogeneous materials, the strain energy $w(\cdot)$ does not depend on the arclength t explicitly when it is written as a function of principal stretches λ and μ. Considering the differentiation $\dot{w}_\lambda = w_{\lambda\lambda}\dot{\lambda} + w_{\lambda\mu}\dot{\mu}$, the resulting equation $(3.3.10)_1$ can be written in the form

$$\dot{r}w_\lambda + r(w_{\lambda\lambda}\dot{\lambda} + w_{\lambda\mu}\dot{\mu}) = -w_\mu \cos\phi. \tag{3.3.11}$$

Using the compatibility equation (3.3.7) to eliminate $\dot{\mu}$, one may get

$$(r\lambda)^{\cdot} = w_{\lambda\lambda}^{-1}\,[(\lambda w_{\lambda\mu} - w_\mu)\cos\phi + (\mu w_{\lambda\mu} - w_\lambda)\,\dot{r}] + \lambda\dot{r} \tag{3.3.12}$$

provided that $w_{\lambda\lambda} \neq 0$. This is ensured by the condition $(1.3.10)_3$. Equation (3.3.12) is particularly useful in treating pole-type singularity at the pole $r = 0$.

Up to this stage, we have not imposed any particular reference configuration and the strain energy function. Thus, the set of equations (3.3.10) are valid for the axisymmetric

deformations of the surfaces of revolution. These equations will be utilized, in particular, in deriving governing equations for wrinkled hemispherical membranes.

Consider the reference geometry (3.3.1) and the non-dimensional scheme (3.3.5), we have

$$r(t) = \cos(t), \quad \dot{r}(t) = -\sin(t). \tag{3.3.13}$$

Substituting (3.3.13) into (3.3.12), (3.3.7) and $(3.3.10)_2$, the governing equations for axisymmetric deformations of a hemispherical membrane with unit radius are

$$\begin{aligned}
\dot{\lambda} &= [\cos(t)\, w_{\lambda\lambda}]^{-1} [(\lambda w_{\lambda\mu} - w_{\mu}) \cos\phi - (\mu w_{\lambda\mu} - w_{\lambda}) \sin(t)] \\
\dot{\mu} &= -\cos(t)^{-1} (\lambda \cos\phi - \mu \sin(t)) \\
\dot{\xi} &= \lambda \sin\phi \\
\cos(t)\, w_{\lambda} \sin\phi &= \tfrac{1}{2} p^{d} \rho^{2} + F^{d}_{c}.
\end{aligned} \tag{3.3.14}$$

where $0 \le t \le \pi/2$. This is a system of equations determining $\dot{\lambda}$, $\dot{\mu}$ and $\dot{\xi}$ in terms of λ, μ and the non-dimensional arclength t. ϕ can be eliminated by substituting $(3.3.14)_4$ into each one of the three preceding equations.

3.4 Inflation of a hemispherical membrane

Before proceeding with the problem involving point load, we first consider the problem of inflation of a hemispherical surface subject to internal pressure alone. The simplest version of the inflation arises from the prescription of the internal pressure intensity. The motivation of undertaking this problem follows Ogden's results [1.22] in which the uniform inflation of an elastic spherical membrane composed of Ogden's material exhibits three possible solutions corresponding to a prescribed pressure. It is worthwhile to note that, for a hemispherical membrane which is fixed at the equator and subject to a uniform internal pressure, there is a certain non-uniqueness in the relation between the pressure and the principal stretch at the pole. More particularly, we have found that, using Ogden's strain energy function defined in (3.1.3), each prescribed value of pressure p^d is associated with three different principal stretch λ_o at the pole. Each λ_o corresponds to a distinct equilibrium configuration. Similar results were also observed by Willson in studying the problem of thin-walled shells [3.7].

Let the membrane in its reference state be a hemispherical surface with a unit radius. The equator is fixed, so that the dimensionless boundary conditions are those specified in (3.3.8). The membrane is assumed to be composed of the isotropic and homogeneous material characterized by the Ogden's strain energy function in (3.1.3). The relevant material constants are given by (3.1.2).

By setting $F^d_c = 0$, the governing equations for the finitely axisymmetric deformations of a hemispherical membrane are given in (3.3.14).

On a close examination of the set (3.3.14), when the reference surface is a complete hemisphere, the pole defined by $r(\pi/2) = 0$ is an isolated singular point in equations

$(3.3.14)_{1,2}$. At the pole, the circumferential stretch $\mu = \rho/r$ is not defined. In particular, we have found that for the hemisphere truncated along a circle of latitude sufficiently near the pole, integration of (3.3.14) will diverge before the integration process arrives at the truncation edge.

This difficulty can be circumvented if we replace equations $(3.3.14)_{1,2}$ by (3.3.12) and $(3.3.6)_1$, respectively, to form a new system of solving equations. The remaining equations of (3.3.14) need not be changed. The concept of this algorithm is to make use of the symmetry condition at the pole to avoid evaluating $(3.3.14)_{1,2}$ at $r = 0$. In solving the axisymmetric inflation of a circular plane sheet, Green and Adkins [1.12] used Taylor's expansion in λ to calculated the λ at a point slightly away from the pole. However, Our method can be regarded as a modified Taylor's expansion, but it does need to use derivatives of higher orders as Green and Adkins did.

By symmetry, at the pole, we have the conditions

$$\lambda = \mu = \lambda_o \text{ (say)}, \quad \phi = 0, \quad \text{at } t = \pi/2 \tag{3.4.1}$$

where λ_o is a constant greater than unity. It is evident that, at $t = \pi/2$, boundary condition $(3.3.8)_3$ holds but $(3.2.10)_2$ is singular.

To consider the pure inflation, we set $F^d_c = 0$ in $(3.3.14)_4$ and integrate equations (3.3.14) numerically. The integration is performed clockwisely from the pole to equator. To do that, we first guess the value λ_o of λ at the pole ($t = \pi/2$) where equation $(3.3.14)_4$ is an identity. Using the initial conditions (3.4.1), equations $(3.3.6)_1$ and (3.3.12) are integrated using a 4^{th}-order Runge-Kutta forward marching scheme from $t = \pi/2$ to $t = \pi/2 - \Delta t$, where Δt is a suitably chosen step size. Instead of λ and μ, integrations of $(3.3.6)_1$ and (3.3.12) yield the values for $r\lambda$ and ρ at $t = \pi/2 - \Delta t$. To

calculate λ and μ, relation $r = \cos(\pi/2 - \Delta t)$ defined by (3.3.1) is used in $(3.2.10)_2$ to obtain μ, while λ is obtained by $\lambda = (r\lambda)/r(\pi/2 - \Delta t)$. This procedure is repeated for each of the mesh point proceeded until the equatorial boundary $t = 0$ is reached. At this point, we calculate the value of ρ. By varying λ_o, we plot ρ at $t = 0$ against λ_o. Comparing the value of ρ at $t = 0$ with that specified by boundary condition $(3.3.2)_1$, the solution of λ_o is found when $(3.3.2)_1$ is satisfied.

As mentioned previously, this integration process was shown to yield multiple solutions of λ_o for each prescribed pressure p. As the matter of fact, we found three λ_o and each one of which defines a distinctive equilibrium state. The task of testing the stability of the equilibrium state associated with each λ_o involves the calculation of second variations of the potential energy (3.3.3). This would be very complicated due to the nature of high nonlinearity of the problem and is beyond the initial scope set for this thesis. We conjecture that the equilibrium state with the smallest potential energy is physically realistic. Nevertheless, other two solutions with higher potential energies also satisfy the necessary conditions of stability presented by Steigmann [1.8]. We compute the potential energy for each λ_o and compare their values to determine the solution. We present these results in Table 3.4.1 for the pressure $p = 1.0, 1.1, 1.2$ that we considered. For each p, λ^i_o, E^i $(i=1,2,3)$ are the respective values of λ at the pole and the associated potential energy.

In particular, Fig.3.4.1 shows the deformed meridian for $p = 1.1$ and $\lambda_o = 1.1540$, 4.185, 6.3825. According to Table 3.4.1, the deformation with the least deflection furnishes the smallest potential energy. These solutions are used as the initial guesses in the problems of pressurized membrane under the action of point-loads.

p = 1.0	$\lambda_o^1 = 1.1303$	$E^1 = -0.033910$
	$\lambda_o^2 = 4.8350$	$E^2 = 1.0811700$
	$\lambda_o^3 = 5.2384$	$E^3 = 1.2096010$
p = 1.1	$\lambda_o^1 = 1.1540$	$E^1 = -0.035324$
	$\lambda_o^2 = 4.1850$	$E^2 = 0.6625420$
	$\lambda_{o3} = 6.3825$	$E^3 = 0.9321834$
p = 1.2	$\lambda_o^1 = 1.1801$	$E^1 = -0.037206$
	$\lambda_o^2 = 3.3075$	$E^2 = 0.3407150$
	$\lambda_o^1 = 7.6384$	$E^3 = 0.7312570$

Table 3.4.1 Solutions of λ_o at the pole and the associated potential energies E.

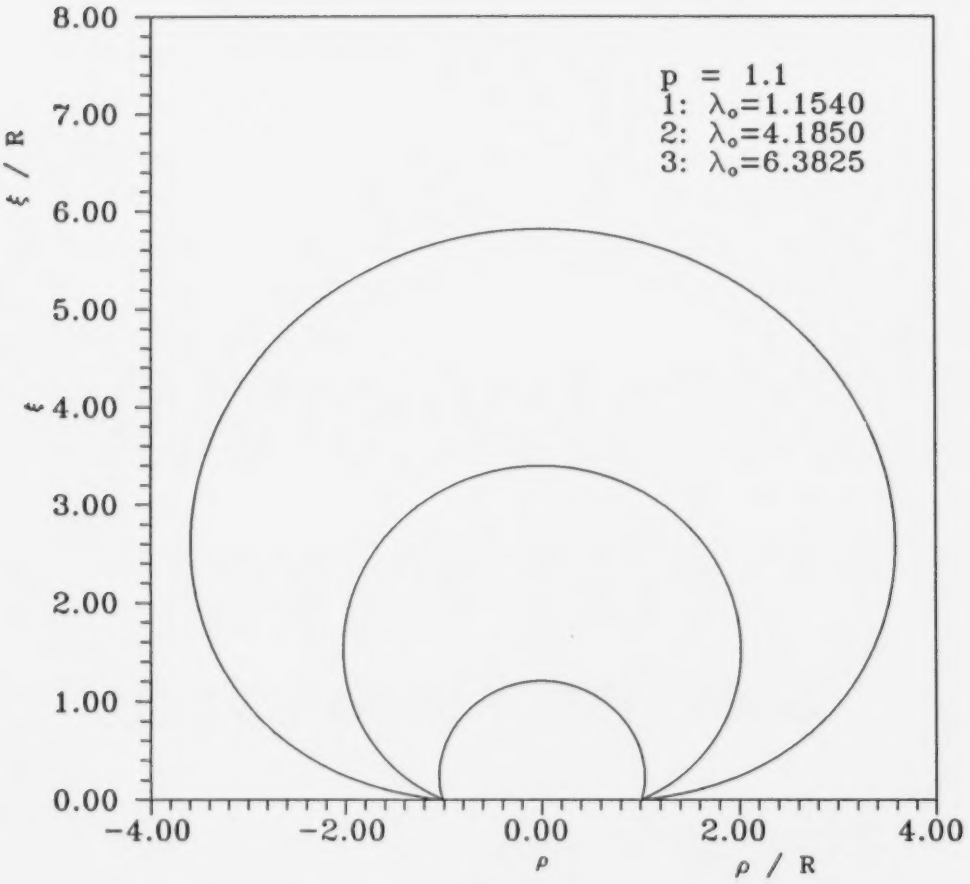

Fig.3.4.1 Profiles of the meridians of the inflated hemisphere

3.5 Point-Load on a Hemispherical Membrane

Analysis in this section provides a unique procedure to produce meaningful singular solutions which are not available by membrane theory in its usual form. This method has been used here to seek the axisymmetric deformations of a hemispherical membrane subjected to a point-load at its pole. In this case, the deformed membrane is partly wrinkled from the pole due to the application of point-loads. Governing equations appropriate to the wrinkled regions are obtained using tension field theory. To do this, the relaxed strain energy of Ogden's material defined by (3.1.7) will be used. Fig.3.5.1 shows a diagram of a radial cross-section of the deformed surface.

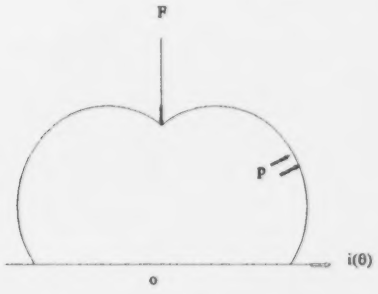

Fig.3.5.1 A deformed hemisphere under point-load

Under certain circumstances, the equilibrium description furnished by the classical membrane theory may be inadequate. One might anticipate difficulties with the results based on these theories. For example, in the point-load problem, in the vicinity of the pole, local meridional stretch becomes infinity, so is the meridional stress for most strain energy functions commonly used. This problem provides an example in which the stress field has a singular point near which the stress is unbounded. The structure of the stress field near such a singular point would be expected to depend on the elastic response of the material at severe stretch. As a result, in a problem involving the same geometry

and loadings, two elastic materials whose constitutive relations are different will generally give rise to quite disparate stress fields near the pole. In terms of the non-linear membrane theory, a hemispherical (spherical) surface subjected to a (pair) point-load(s) at its pole(s) is the representative example of this class. Related references may be found in [3.2,8,9]. The objective of this section is to establish a quadrature by which the solution for extensible membrane surfaces of revolution undergoing the axisymmetric deformations can be obtained.

Previously, some approximate solutions have been found for elastic membranes involving point-loads. To avoid the difficulty caused by the singularities in the stress field near the pole, solutions were obtained by assuming either the material to be inextensible in the neighbourhood of the point of application of the point-load, e.g. Libai [3.8], or the entire membrane is assumed to be inextensible, e.g. Szyszkowski and Glockner [3.9].

In this work, we present an analytic solution to a partly wrinkled elastic membrane involving point-load. The membrane is assumed to be extensible and is made of Ogden's material whose strain energy function adapted for use in the membrane theory is defined by (3.1.3). As demonstrated later, not every strain energy function leads to the solvability of the governing equations. This fact was previously pointed out by Fulton [3.2] in examining the problem of a circular membrane subject to an axisymmetric deflection through a pull applied on the central rigid hub. It was shown that, as the hub radius becomes small, the solution with the finite deflection at the hub depends on the functional form of the strain energy function of the material of which the membrane is made. In this section, for the hemispherical elastic membrane, we are able to

demonstrate that the existence of solutions with finite deflection at the point of application of the point load requires a certain growth condition to be satisfied by strain energy functions involved. As shown in Appendix B, this condition is fulfilled by Ogden's strain energy function (3.1.9), but not by several of the strain energies of engineering importance commonly used in applications of membrane theory. In particular, for instance, such solutions do not exist for the materials whose strain energy is composed of quadratic terms in λ and μ such as the neo-Hookean material used by Fulton [3.2]. Although the example we consider in this section is for the hemispherical surfaces, the theory itself can be readily extended to the wrinkling analysis of surfaces of revolution when they are axisymmetrically deformed by dead point-loads.

For a hemispherical membrane subjected to point-loads and fixed at the equator, the deformed surface is partly wrinkled in an area near the pole. To prove this assumption qualitatively, properties of the relaxed strain energy (1.3.26) may be utilized to demonstrate that the membrane is tense near the equator. As a deformation hypothesis, we assume that the deformed membrane is tense in the region $t \in [0, t_b]$ adjacent to the equator, where t_b is used to denote the boundary between the wrinkled and tense regions. Suppose the assumption is not true and the entire membrane is wrinkled for the given loading parameters. The deformed surface is a meridional tension field. In the wrinkled region, the relaxed strain energy $w^R(\lambda)$ must be used in governing equations (3.3.14). In view of the boundary condition $(3.3.8)_1$, the circumferential stretch $\mu = 1$ at the equator, where $\rho = \mu$ comes from $(3.3.5)_3$. Thus the right-hand side of equations $(3.3.14)_4$ is non-zero at $t = 0$, i.e. $(F^d_c + \frac{1}{2}p^d) \neq 0$. We consider the case for which $(F^d_c + \frac{1}{2}p^d) > 0$. It is clear that this condition is satisfied strictly if $F^d_c > 0$, i.e. the

point-load is applied upwardly at the pole. In the case that the point-load is applied downwardly, $F^d_c < 0$, values of F^d_c must be restricted to the range $[-\frac{1}{2}p, 0]$. If all these conditions are satisfied, according to $(3.3.14)_4$, the uniaxial stress resultant w^R_λ is non-zero and positive. This implies $\lambda > 1$ since current state of stretch corresponds to $(1.3.26)_2$. The state of wrinkling requires that $\mu < v(\lambda) = \lambda^{-1/2} < 1$ for $\lambda > 1$ at the equator. The corresponding state of μ is contradictory to $(3.3.8)_1$. Therefore, by continuity, the membrane is tense up to $t = t_b$, for some $t_b > 0$.

As mentioned previously, for the axisymmetric deformations of a membrane surface of revolution under the action of point-loads, the governing equations in the wrinkled region may be further simplified using the special features of relaxed strain energy function $w^R(\lambda)$. As shown later, the system of governing equations is reduced to a quadrature in wrinkled regions. This special reduction is not applicable to the membrane in tense state. To distinguish this special characteristics, we would discuss the governing equations in the two kinds of regions separately.

We first demonstrate the reduction of the governing equations to a quadrature in the wrinkled region. It is seen from the boundary condition $(3.3.8)_3$ that the deformed radius $\rho(t) \to 0$ as $t \to \pi/2$. Suppose p^d, F^d_c in equation $(3.4.14)_4$ are constants, as $\rho \to 0$, the right-hand side of $(3.3.14)_4$ approaches the constant F^d_c. In the meantime, on the left-hand side, $r \to 0$, this is compensated for by $w^R_\lambda \to \infty$. For the relaxed strain energy function (3.1.9) of Ogden's material, this implies that, in the neighbourhood of the pole, the meridional stretch λ is also unbounded. Usually, the typical relaxed strain energy functions used in practice exhibit the analogously asymptotic behaviours. Based on this, we speculate that $(3.3.14)_2$ yields such $\dot{\mu}$ that $\dot{\mu} \to -\infty$ near the pole, where $\dot{\mu}$ is the

derivative of μ with respect to arclength t. This implies that μ decreases rapidly as $t \to \pi/2$. Zak [1.5] has examined a hemispherical surface pulled apart axisymmetrically by a point-load alone at the pole. By setting the circumferential stress resultant to zero in the equilibrium equations, he demonstrated that even an infinitesimal amount of pulling at the pole will result in a finite value in radial contraction of the reference meridians. The deformed membrane is wrinkled circumferentially over the entire surface and the deformed surface forms a cone of revolution. Zak's solutions were obtained without reference to the properties of the material of the membrane. However, Zak's solutions are erroneous since More recently, using the elastic model, Libai [3.2] considered the same problem with particular attention to the tense region near the equator. The solution of the deformed surface is a circular cone abutted by a narrow band of tense zone adjacent to the rigid equator. Following the qualitative behaviours of Zak and Libai's solutions, we speculate for the present that the membrane is circumferentially wrinkled in the interval $[t_b, \pi/2]$, i.e. from the point t_b near the equator up to the pole. This assumption is verified a posteriori.

According to its definition, the relaxed strain energy $w^R(\lambda)$ is independent of the circumferential stretch μ in the wrinkled region $[t_b, \pi/2]$. It leads to the fact that $w^R_{\mu} = w^R_{\lambda\mu} = 0$. Thus in the wrinkled region tensile stresses exist only in the meridional direction, while the tension in the circumferential direction vanishing identically. It is noted that $w^R_{\mu} = 0$ in $(3.3.10)_1$ results in the first integral in the wrinkled region,

$$r(t)w^R_{\lambda}(\lambda) = c, \tag{3.5.1}$$

where c is an integration constant to be determined. Relation (3.5.1) is obtained without reference to the geometries of reference and deformed surfaces. This follows the fact

that stresses in the wrinkled regions are determined independent of the deformation, i.e. they are statically determinate. Substituting (3.5.1) into (1.4.1) for w^R_λ, we have the Piola stress tensor $\underline{T} = (c/r)\, \underline{1} \otimes \underline{L}$. In view of (3.2.11), it determines the distribution of a uniaxial stress field $w^R_\lambda(\cdot)$ in the meridional direction conclusively once the constant c is specified. It is also evident that (3.5.1) is obtained regardless of the loadings and the elastic properties of the membrane materials. Therefore it is a relation valid for axisymmetric deformations of the wrinkled surfaces of revolution made of arbitrary elastic material. Loading and reference geometry influence the solutions through the constant c. For example, if the reference surface is a circular cylinder, i.e. r = R (const), (3.5.1) indicates that λ remains as a constant in the wrinkled region regardless of the shape of the deformed surface. This behaviour is shown in a problem considered later in section 3.9. Furthermore, solution (3.5.1) allows the insight observation of the singular behaviour of stress in the neighbourhood of the pole. In view of equation $(3.3.14)_4$, as the deformed radius ρ passes to zero, it is clearly shown that such solutions require the presence of, at least, a point-load F^d_c in order to maintain the equilibrium. In other word, the internal pressure p^d alone causes no wrinkling on the deformed surface.

Considering the fact that the relaxed strain energy $w^R(\lambda)$ is uniquely determined by the non-vanishing principal stretch λ alone, equation (3.5.1) may be regarded as a relation by which functions of t can be transformed to functions of λ. As shown later in the section, it is crucial to use λ, instead of t, as the independent variable to reduce the governing equations to a quadrature. To proceed with the variable transformation, we first differentiate (3.5.1) to obtain the equation

$$dr = - rf'(\lambda)/f(\lambda)\, d\lambda \tag{3.5.2}$$

where $f(\lambda) = w^R{}_\lambda(\lambda)$ is the strain-stress relation in uniaxial tension. The prime denotes the differentiation with respect to λ.

Specifically, considering the geometrical relation (3.3.13), we have

$$dr = - (1 - r^2)^{1/2}\, dt \tag{3.5.3}$$

Here the minus sign is restrictive for the hemispherical reference surface specified by (3.3.1).

On substitution of (3.5.3) into (3.5.2) and using (3.5.1) to eliminate r, we have the differential relation

$$dt = \frac{c\ f'(\lambda)}{f(\lambda)\sqrt{f^2(\lambda) - c^2}}\, d\lambda. \tag{3.5.4}$$

Note that in the interval $t \in [t_b, \pi/2]$, $0 < r(t) < 1$, thus $f(\lambda)$ obtained from (3.5.1) ensures the inequality of $f^2(\lambda) - c^2 > 0$. From (3.1.10) and (3.1.11), we have $f(\lambda) > 0$ and $f'(\lambda) > 0$ for $\lambda > 1$. Considering these conditions and the fact that the integration constant c is non-zero and positive as shown by (3.5.1), we conclude that the coefficient of $d\lambda$ in (3.5.4) is strictly non-zero and positive. This implies that the derivative $dt/d\lambda$ is a monotonically increasing function of λ everywhere in the interval $t \in [t_b, \pi/2]$. The monotonical relation thus ensures that the change of independent variable from t to λ through (3.5.1) is one-to-one in the range $t \in [t_b, \pi/2]$.

Substituting (3.5.1) back into $(3.3.14)_4$ determines the angle ϕ as a function of ρ explicitly:

$$\sin \phi = (\tfrac{1}{2}\, p^d \rho^2 + F^d{}_c)/c. \tag{3.5.5}$$

Apparently (3.5.5) is independent of the reference geometry and material property of the membrane. If, for instance, pressure $p^d = 0$ and $F^d_c = \text{const.}$, $\sin\phi$ becomes a constant, thus the deformed surface for the point-load F^d_c alone is conical in the wrinkled region. This coincides with Zak's result [3.14].

To proceed further, we need expression of $\cos\phi$ in term of ρ. We make the additional assumption that $\cos\phi > 0$ everywhere in the wrinkled region $t \in [t_b, \pi/2]$. Then $\cos\phi$ can be uniquely determined from (3.5.5) by

$$\cos\phi = [c^2 - (\tfrac{1}{2} p^d\rho^2 + F^d_c)^2]^{1/2}/c. \qquad (3.5.6)$$

We may as well examine the possibility of $\cos\phi = 0$. For the problem under consideration, $\cos\phi = 0$ is possible only at the pole ($\rho = 0$). For the case that $\cos\phi = 0$ everywhere, the corresponding meridian of the deformed surface becomes a straight line along the axis of symmetry $\underline{k}$. This would never occur for the deformations in which the equator is fixed, whereas it does occur when the equator is free of constraints. We consider the latter case as a special example in detail in section 3.7. We now examine the effects of $\cos\phi = 0$ locally, setting $\rho = 0$ in (3.5.6), $\cos\phi = 0$ results in $c^2 = (F^d_c)^2$. Since $c > 0$, there exists a pair of $F^d_c = |F^d_c|, -|F^d_c|$ satisfying the equation. If $F^d_c > 0$, the first branch $F^d_c = |F^d_c|$ is valid, i.e. $\underline{F}^d_c$ is applied upwardly. According to the assumption preceding equation (3.5.6), $\cos\phi > 0$ ensures that the square root is in the real-value domain. This condition is strengthened by the strict inequality $[c^2 - (F^d_c + \tfrac{1}{2} p^d\rho^2)^2] > 0$. This is obtained from our deformation hypothesis of $\mu < 1$ for $t > t_b$. If $p^d \geq 0$, it indicates that $[c^2 - (F^d_c)^2] > 0$ for $t > t_b$. Thus, we conclude that the assumption of $\cos\phi > 0$ is valid for all values of $F^d_c \geq 0$ and $p^d \geq 0$. This also implies that the membrane in the neighbourhood of the pole will never

collapse onto each other (see Fig.3.5.2). If $F^d_c < 0$, we only consider the range of pressure p for which $\cos\phi = 0$ ($\phi = -\pi/2$) occurs only at the pole. However, values of F and p for which $\phi = -\pi/2$ must be calculated numerically. When $\phi = -\pi/2$, the neighbourhood of the membrane near the pole starts collapsing onto each other along the axis $\underline{k}$. Our theory fails to account for the deformation when p^d is beyond this value. This limiting case is illustrated in the numerical examples displayed later.

Substituting (3.5.4) into the compatibility equation $(3.3.6)_1$, we obtain a transformed differential equation,

$$\frac{d\rho}{\cos\phi} = - \frac{c\,\lambda f'(\lambda)d\lambda}{f(\lambda)\sqrt{f(\lambda)^2 - c^2}}, \tag{3.5.7}$$

Substituting (3.5.6) for $\cos\phi$ into (3.5.7), the left-hand side becomes the function of ρ only, whereas the right-hand side is only the function of λ. Therefore two sides of (3.5.7) can be integrated independently once the appropriate upper and low limits are specified. They are associated with boundary conditions (3.3.8). To integrate (3.5.7), $f(\lambda)$ and $f'(\lambda)$ are replaced by expressions $(3.1.10)_1$ and $(3.1.11)_1$, respectively. It is noticeable that (3.5.7) is obtained with particular reference to the hemispherical surface specified by (3.3.1).

To complete the governing equation reduction, we write equation (3.5.7) in an integral form with the boundary conditions (3.3.8). Based on our hypothesis, the deformed membrane is wrinkled and $\cos\phi > 0$ in the interval $[t_b, \pi/2]$, using ρ_b and λ_b to denote the limiting values of ρ and λ as $t \to t_b^+$, where t_b^+ is the limiting point as t approaches t_b from the wrinkled side. From $(3.3.8)_3$, $\rho \to 0$ as $t \to \pi/2$. In accordance with $(3.3.14)_4$, uniaxial stress-stretch relation $(3.1.10)_1$ yields $\lambda \to \infty$ as $t \to \pi/2$. These

values provide the upper limits for integral (3.5.7). Substituting (3.5.6) into the left side of (3.5.7) to replace $\cos\phi$, we are able to integrate (3.5.7) from the interior boundary point t_b to the pole

$$\int_{\lambda_b}^{\infty} \frac{\lambda f'(\lambda)d\lambda}{f(\lambda)\sqrt{f(\lambda)^2 - c^2}} + \int_{\rho_b}^{0} \frac{d\rho}{\sqrt{c^2 - (F^d{}_c + \tfrac{1}{2} p^d \rho^2)^2}} = 0, \qquad (3.5.8)$$

provided that the integrals exist. Assumption of $\cos\phi > 0$ for $t \in [t_b, \pi/2]$ ensures the existence and integrability of the second integral. At the interior boundary point $t = t_b$ ($< \pi/2$), $r_b = r(t_b) = \cos(t_b) > 0$, thus the value λ_b of the meridional stretch λ obtained from (3.5.1) is finite. Consequently, the integrand of the first integral evaluated at t_b is also finite. The uniaxial stress-stretch relation $f(\lambda)$ ($=w^R{}_\lambda$) specified by $(3.1.10)_1$ consists of a series of powers in λ, the dominant term in the asymptotic function is λ^{p-1} ($p>1$) as $\lambda \to \infty$. Substitute $(3.1.10)_1$, $(3.1.11)_1$ in the integrant of the first integral in (3.5.8), it is found that the asymptotic function of the first integrand is $\lambda f'(\lambda)/f^2(\lambda)$ as $\lambda \to \infty$. Thus existence is assured if $\lambda f'(\lambda)/f^2$ decays faster than λ^{-1} as $\lambda \to \infty$. According to $(3.1.10)_1$, this requires that the largest exponent of λ in $f(\cdot)$ must be such that $p^d > 1$. Thus a solution to the point-load problem exists provided that $f(\cdot)$ is superlinear when λ is large. Replacing $f(\lambda)$ and $f'(\lambda)$ in the first integrand by $(3.1.10)_1$ and $(3.1.11)_1$, it is seen that this *Growth Condition* is fulfilled by Ogden's relaxed strain energy with elastic constants specified by (3.1.2) (see Appendix B). However, this condition fails for the Hookean, Neo-Hookean and Mooney-Rivlin strain energies in which the linear term of λ in $f(\lambda)$ prevails as $\lambda \to \infty$. A similar observation was made by Steigmann [3.2] in connection with the problem of a closed surface of revolution with point loads applied at the apices. These conclusions qualitatively agree with Fulton's analysis [3.12].

Equation (3.5.8) can also be used to determine the deformed radius ρ in terms of $\lambda(t)$ in the wrinkled region $t \in [t_b, \pi/2]$. To compute $\rho(t)$ at interior points $t \in (t_b, \pi/2)$, we first obtain $\lambda(t)$ from (3.5.1), then we replace the upper limits in the first and second integral in (3.5.8) by $\lambda(t)$ and ρ, respectively. The shape of the deformed meridian can be obtained by integrating ξ in $(3.3.6)_2$. To see this, we write the compatibility equation $(3.3.6)_{1,2}$ in the form

$$\frac{d\rho}{d\xi} = -\tan\phi \tag{3.5.9}$$

In view of the boundary conditions (3.3.8), we combine (3.5.5) and (3.5.6) to express $\tan\phi$ in terms of ρ. The value of $\xi(t)$ at an interior point t can be calculated by integrating (3.5.9) from the interior boundary t_b to the point of interest,

$$\xi(t) = \xi_b - \int_{\rho_b}^{\rho(t)} \frac{(F^d{}_c + \tfrac{1}{2} p^d \rho^2)d\rho}{\sqrt{c^2 - (F^d{}_c + \tfrac{1}{2} p^d \rho^2)^2}}, \tag{3.5.10}$$

where $\rho(t)$ is the solution of (3.5.8) and $\rho(t) < \rho_b$. ρ_b and ξ_b are values of ρ and ξ at t_b. If the internal pressure $p^d = 0$, the integrant becomes a constant and (3.5.10) yields a circular cone in the wrinkled region.

In the interval $[0, t_b]$, the membrane is tense. Governing equations (3.3.14) are operative. The availability of boundary conditions at the equator $t = 0$ makes it possible to treat the system of equations (3.3.14) as an initial value problem for the first order differential equations in λ, μ and ϕ.

Proof of existence of the foregoing solutions necessitates the need to demonstrate that the solutions we have obtained furnish finitely-valued potential energy. In this effort, we

write the total potential energy as a sum of integral of (3.3.3) over the tense and wrinkled regions. Based on our deformation hypothesis, (3.3.3) is the sum

$$E/2\pi GR = \int_0^{t_b} \left[r(t)w(\lambda,\mu) - \frac{1}{2}p^d\rho^2\dot{\xi} \right] dt + \int_{t_b}^{\pi/2} \left[r(t)w^R(\lambda) - \frac{1}{2}p^d\rho^2\dot{\xi} \right] dt - F^d{}_c\xi(\frac{\pi}{2}) \tag{3.5.11}$$

The first and second integral accounts, respectively, for the total strain energy in the tense and wrinkled regions. Since λ and μ are continuous in the tense region $[0, t_b]$, and finite at $t = 0$ and $t = t_b$, by continuity of the solution, the value of the first integral in (3.5.11) is thus proved to be bounded. It remains to verify that the total strain energy associated with the wrinkled region is also bounded. To establish this, we substitute (3.5.4) and $(3.3.6)_2$ into the second integral in (3.5.11) for dt and $\dot{\xi}$ to express the total strain energy in wrinkled region in the form (noting that $\lambda \to \infty$ as $t \to \pi/2$),

$$E_w/2\pi RG = c \int_{\lambda_b}^{\infty} \left[\frac{c\,w^R(\lambda)}{\lambda\, f(\lambda)} - \frac{1}{2}\, p^d\rho^2 \sin\phi \right] \left[\frac{\lambda\, f'(\lambda)}{f(\lambda)\sqrt{f(\lambda)^2 - c^2}} \right] d\lambda\,, \tag{3.5.12}$$

We have proved previously that the function in the second square bracket has an asymptotic function $\lambda f'(\lambda)/f^2(\lambda)$ as $\lambda \to \infty$, and this term approaches zero if the leading term λ^{p-1} in $f(\lambda)$ has the exponent such that $p > 1$. This growth condition is fulfilled by $f(\lambda)$ associated with Ogden's strain energy function $(3.1.10)_1$. Following the same analysis, substituting (3.1.10) and (3.1.11) into (3.5.12), it is found that the first term in the first square bracket tends to be the constant c as $\lambda \to \infty$. Since $0 < |\sin\phi| < 1$ and p^d is a constant, the second term in the first square bracket approaches zero as $\rho \to 0$ ($\lambda \to \infty$). Therefore, the integrand of (3.5.12) is finite at t_b and asymptotically

decreases to 0 as $\lambda \to \infty$. Also it is continuous in the domain $[\lambda_b, \infty)$. Thus the existence of a finite E_w is secured for Ogden's material.

Deformations for the point-load have been considered with and without the internal pressure, respectively.

We first consider the point-load F^d_c alone acting upwardly at the pole. To obtain the solution, we first guess the value λ_o (> 1) of λ at the equator $t = 0$, while $\mu_o = 1$ is given by the boundary condition $(3.3.8)_1$, using these values in $(3.3.14)_4$ ($p^d = 0$), the angle ϕ_o is calculated. Next, the set of equations (3.3.14) ($p^d = 0$) in tense region are integrated using a Runge-Kutta forward-marching scheme from $t = 0$ to $t = \Delta t$, where Δt is a suitably chosen step size. The values of λ, μ, ϕ and ζ are then updated. ρ in $(3.3.14)_4$ can be calculated using compatibility equation $(3.2.10)_2$, i.e. $\rho = r(t)\mu$. This procedure is repeated for each of the mesh point proceeded until the interior boundary point $t = t_b$ is encountered. At t_b, condition $\mu_b = \upsilon(\lambda_b) = \lambda_b^{-1/2}$ is imposed. Due to the assumption that the deformation is of class C^1, λ, μ, ϕ, ρ and ξ are continuous at $t = t_b$. As a result, values $\rho_b = \rho(t_b)$, $\lambda_b = \lambda(t_b)$ and $\zeta_b = \zeta(t_b)$ obtained in tense region can be used to provide the low limits for the integral (3.5.8) and (3.5.10). These values are functions of the initial guess λ_o implicitly,.

Following the deformation hypothesis that the membrane is wrinkled for $t > t_b$, we use λ_b obtained by tense solution to determine the constant $c = r(t_b)f(\lambda_b)$, $\rho_b = r(t_b)\mu_b$. The values of λ_b, ρ_b and c are then substituted into the right hand side of equation

$$V(\lambda_0) = \int_{\lambda_b(\lambda_o)}^{\infty} \frac{\lambda f'(\lambda)\, d\lambda}{f(\lambda)\sqrt{f(\lambda)^2 - c^2}} + \int_{\rho_b(\lambda_o)}^{0} \frac{d\rho}{\sqrt{c^2 - (F^d_c + \tfrac{1}{2} p^d \rho^2)^2}}, \qquad (3.5.13)$$

in which we have replaced the lower limits λ_b and ρ_b by $\lambda_b = \lambda_b(\lambda_o)$ and $\rho_b = \rho_b(\lambda_o)$ in order to emphasize the dependence of λ_b and ρ_b on the initial guess λ_o. Equation (3.5.13) are evaluated numerically using Simpson's integration algorithm. By varying the initial guess λ_o, values of function $V(\lambda_o)$ may be plotted against λ_o and the root λ_o of $V(\lambda_o) = 0$ can be easily found. Solution λ_o corresponds to an equilibrium state. This procedure has been found to yield a unique λ_o for each set of the loading parameters that we have considered. For the solution λ_o obtained, the shape of the deformed meridian in the wrinkled region may be obtained from (3.5.10). To evaluate the first integral in (3.5.13), we make the change of variable $\lambda \to x^{-1}$ and integrate from $x = 0$ to $x = \lambda_b^{-1}$.

Table 3.5.1 presents the solutions of λ_o for $F^d_c = 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40$. t^i_b $(i=1,2,..7)$ denotes the locations of the boundary point separating the tense and wrinkled regions. Values of t^i_b demonstrate the extent of the tense layer near the equator and is seen to increase with increasing forces. This is because the increase in λ consequently reduces the associated natural width $\upsilon(\lambda)$. This results in the recovery of the wrinkled part back to the tense state. This result is in qualitative agreement with Libai's analysis [3.2]. Fig.3.5.2. shows the corresponding profiles of the deformed meridians. It reveals that, for the values of F^d_c we have considered, the angle ϕ between the inward radial axis and the tangent to a meridian is such that $\cos\phi > 0$. This is in accordance with the assumption we made. The distributions of the meridional stretch $\lambda(t)$ are plotted in Fig.3.5.3 as the function of elevation angle t. It is found that values of λ are unbounded near $t = \pi/2$ (pole). In each case a little more than half of the solution is shown. For the part not shown, λ deviates only slightly from its value λ_o at the equator $t = 0$.

We must also verify the assumption that the membrane is tense near the equator and wrinkled elsewhere. We use solutions of λ_o and the boundary conditions (3.3.8) at equator to integrate the equations (3.3.14). For the points in tense region, λ is calculated by $(3.3.14)_1$, whereas in wrinkled region, λ is obtained by simply solving the algebraic equation (3.5.1). It is found that near the pole ($t \to \pi/2$), $(3.3.14)_2$ exhibits strongly singular behaviour, but the integration can be extended to a point very near the pole if a sufficiently fine mesh is used. The results are plotted in Fig.3.5.4 in the form of $\mu\lambda^{1/2}$ against the elevation angle t. It is shown that the membrane is tense wherever $\mu\lambda^{1/2} > 1$, while wrinkled when $\mu\lambda^{1/2} < 1$. The point at which $\mu\lambda^{1/2} = 1$ indicates the boundary between the tense and wrinkled regions. This is in accordance with the definition of relaxed strain energy and our hypothesis is proved to be valid.

Discontinuities in $\dot{\lambda}$ and $\dot{\mu}$ are expected by the use of relaxed strain energy $w^R(\lambda)$. We infer this because the strain energy $w(\cdot)$ does not possess the C^2-continuity on the deformation gradient $\underline{F}$.

This becomes evident when the relaxed strain energy function is used in the wrinkled region. By substituting (3.1.8) into (3.1.4), it shows that the limiting values of w_λ at $t = t_b$ is continuous, namely, $w_\lambda(\lambda, \upsilon(\lambda)) = w^R{}_\lambda(\lambda)$. However, second derivatives $w_{\lambda\lambda}$ and $w_{\lambda\mu}$ possess discontinuity of finite jumps at t_b. The limiting values of $w_{\lambda\lambda}$ and $w_{\lambda\mu}$ at t_b approached from the tense-region side are calculated by setting $\mu = \upsilon(\lambda)$ in (3.1.6) and (3.1.7), respectively. We have

$$w^-{}_{\lambda\lambda}(\lambda, \nu(\lambda))\big|_b = \sum_{p=1}^{3} g_p(\alpha_p - 1)\lambda_b^{(\alpha_p - 2)} + (\alpha_p + 1)\lambda_b^{-(\frac{\alpha_p}{7} + 2)} \qquad (3.5.14)$$

$$w^{-}{}_{\lambda\mu}(\lambda, \nu(\lambda))|_b = \sum_{p=1}^{3} g_p \alpha_p \lambda_b^{-\frac{(\alpha_p+1)}{2}} \tag{3.5.15}$$

On the other hand, the limiting values of $w_{\lambda\lambda}$ and $w_{\lambda\mu}$ at $t = t_b$ approached from the wrinkled region side are evaluated by

$$w_{\lambda\lambda}{}^{+} = w^{R}{}_{\lambda\lambda}|_b = \sum_{p=1}^{3} g_p(\alpha_p - 1)\lambda_b^{\alpha_p - 2} + (\frac{1}{2}\alpha_p + 1)\lambda_b^{-(\frac{\alpha_p}{2}+2)} \tag{3.5.16}$$

$$w^{+}{}_{\lambda\mu} = w^{R}{}_{\lambda\mu}|_b = 0 \tag{3.5.17}$$

Subtracting expressions (3.5.14), (3.5.15) from (3.5.16), (3.5.17), respectively, we obtain the values of finite jump discontinuity in $w_{\lambda\lambda}$ and $w_{\lambda\mu}$ at t_b. They are

$$[\![w_{\lambda\lambda}]\!]_b = w^{+}{}_{\lambda\lambda}|_b - w^{-}{}_{\lambda\lambda}|_b = \frac{1}{2}\sum_{p=1}^{3} g_p \alpha_p \lambda_{t_b}^{-(\frac{\alpha_p}{2}+2)} \tag{3.5.18}$$

$$[\![w_{\lambda\mu}]\!]_b = w^{+}{}_{\lambda\mu}|_b - w^{-}{}_{\lambda\mu}|_b = \sum_{p=1}^{3} g_p \alpha_p \lambda_{t_b}^{-\frac{(\alpha_p+1)}{2}} \tag{3.5.19}$$

in which jumps $[\![w_{\lambda\lambda}]\!]_b$ and $[\![w_{\lambda\mu}]\!]_b$ do not vanish unless $\lambda_b = 0$. The state of deformation for which $\lambda_b = 0$ corresponds to the meridional wrinkling in the neighbourhood of t_b. This corresponds to a slack state according to $(1.3.26)_4$. This contradicts the assumption of C_1-continuity possessed by the deformed meridional curves. We exclude this state of deformation from our solutions. It is concluded that discontinuities $[\![w_{\lambda\lambda}]\!]_b$ and $[\![w_{\lambda\mu}]\!]_b$ exist if relaxed strain energy is used.

In wrinkled region, the strain energy function is replaced by relaxed energy $w^R(\lambda)$ which is dependent on λ alone, then $w^R{}_{\mu} = w^R{}_{\lambda\mu} = 0$. Noting that $w^R(\cdot)$ does not depend on the variable t explicitly, we can write equation $(3.5.14)_1$ for the wrinkled

region. Thus the limiting value of $\dot{\lambda}$ at $t = t_b$ approached from the wrinkled-region side is

$$\dot{\lambda}_b^{+} = - [w^R{}_{\lambda}\dot{r}/ (rw^R{}_{\lambda\lambda})]_b, \tag{3.5.20}$$

Approached from the tense-region side, the limiting value of $\dot{\lambda}$ at $t = t_b$ ($u_b = \upsilon(\lambda_b)$) is determined by using $(3.3.14)_1$

$$\dot{\lambda}_b^{-} = (rw_{\lambda\lambda})_b^{-1}[(\lambda w_{\lambda\mu} - w_{\mu}) \cos \phi + (\mu w_{\lambda\mu} - w_{\lambda})\dot{r}]_b, \tag{3.5.21}$$

Subtracting (3.5.20) from (2.5.21) yields the value of jump discontinuity in $\dot{\lambda}$ at $t = t_b$. It is

$$\begin{aligned} [\![\dot{\lambda}]\!]_b &= \dot{\lambda}_b^{+} - \dot{\lambda}_b^{-} \\ &= (rw_{\lambda\lambda})_b^{-1}[(\lambda w_{\lambda\mu} - w_{\mu}) \cos \phi + (\mu w_{\lambda\mu} - w_{\lambda})\dot{r}]_b + [w^R{}_{\lambda}\dot{r}/ (rw^R{}_{\lambda\lambda})]_b \end{aligned} \tag{3.5.22}$$

which, clearly, does not vanish at $t = t_b$ due to the non-vanishing jumps $[\![w_{\lambda\lambda}]\!]_b$ and $[\![w_{\lambda\mu}]\!]_b$. As shown in (3.5.22), the jump is proportional to $w_{\lambda\mu}$. These discontinuities occurred in our calculations and are displayed graphically in Fig.3.5.4 at the boundary point $t = t_b$.

Secondly, we consider the point-load deformations involving internal pressures. Numerical examples are presented for a pressurized hemispherical membrane ($p^d = 1.0$) subjected to the successive application of a series of the downwardly-directed point forces ($F < 0$) at the pole. Solutions of λ_o for $p = 1.0$ and various values of $F^d{}_c$ are presented in Table 3.5.2. To serve as the reference, the deformation for the internal pressure alone ($F^d{}_c = 0$) is plotted together with those deformations associated with the point-load. Fig.3.5.5 shows the profiles of the deformed meridians for successive stages of application of the point-load $F^d{}_c$. The heavy dots are used to indicate the interior boundaries between the tense and wrinkled regions. It is evident that $\cos \phi > 0$ is

satisfied in the wrinkled region, thus our assumption is verified a posteriori. Fig.3.5.5 shows the fact that wrinkling starts near the pole for the very small forces and grows progressively with increasing force toward the equator until it covers nearly the entire deformed surface. However, the deformed membrane will never wrinkled entirely, and the wrinkled region is always abutted by a tense layer near the rigid equator. The size of the tense layer varies depending on the loading conditions. This behaviour is in qualitative agreement with the analysis of Szyszkowski and Glockner [3.13] for inextensible membranes. To avoid crowding, the distributions of $\lambda(t)$ are shown in Fig.3.5.6 when the values of F^d_c = -0.01, -0.05, -0.10, -0.16, -0.18, -0.24, -0.26. As is clearly shown in Fig.3.5.6, for F^d_c = -0.26, the tangent of the deformed membrane at the pole starts to become vertical, the central angle at the point-load, $\phi \rightarrow -\pi/2$, and the integral in (3.5.8) become divergent. This indicates that the maximum value of F^d_c is nearly achieved for the pressure p^d = 1.0. The plot of $\mu\lambda^{1/2}$ against t in Fig.3.5.7 shows the distinctive regions. $\mu\lambda^{1/2} < 1$ indicates that the membrane is wrinkled, while $\mu\lambda^{1/2} > 1$ indicates the membrane is tense. Again, it verifies our wrinkling hypothesis. It is appropriate to interpret the solutions obtained for $F^d_c < 0$ as members of a sequence tending to the full tense solution for small F^d_c.

Attempt was also made to find point-load solutions near the two higher-energy, zero-force solutions (see Table 3.4.1. λ^2_o = 4.8350, λ^3_o = 5.2384 at p^d =1.0). In these cases, for whatever initial guess λ_o we have tried, numerical integration based on (3.3.14) diverged near the pole before any wrinkling could be detected.

For the solutions obtained, we have computed the associated Legendre-Hadamard conditions (1.3.10) and (1.3.11). They are found to be satisfied at all mesh points in the

region where the membrane is tense. As indicated in section 1.3, these conditions are automatically satisfied in wrinkled regions. Since they are only the necessary conditions for local stability, the stability of the computed deformations remains as an open question. However, as shown in Fig.3.5.5, the relation between the point-load F^d_c and the vertical pole deflection, namely ξ_{pole}, presents a one-to-one correspondence. There is no presence of limit-point or snap-through behaviour in the range of loading parameters p^d and F^d_c that we have considered. To confirm the observation, we calculated the solution for the pressure $p^d = 1.3$. The F^d_c - ξ_{pole} relation was shown to exhibit analogous behaviours. For the hemispherical membrane made of inextensible material, Szyszkowski and Glockner [3.15] have shown that snap-through behaviour does not occur, but does occur if the elevation angle t has a range that exceeds $\pi/2$. Guided by this, we predict that our solutions for the values of p and F are stable at least in the sense of local infinitesimal stability.

Because the solution-procedure demonstrated above may seem to appear in relatively complicated form, one may question the scope of its applicability. Immediately, it may be observed that this method can be applied in the case, among others, that the point-load F^d_c is replaced by prescribing the axial displacement ξ_{pole} at the pole. Solvability of the problem by this method lies in the fact that numerical values of sufficient number of variables must be available at the common boundaries to provide upper and low limits for the integral (3.5.8). Unfortunately, it is not always so. For example, this method fails to work for such deformation in which the deformed membrane is wrinkled in an interior region. Under this circumstance, as shown previously, we are only able to calculate values of the variables needed at one of the interior boundary, whereas, values

of the variables at the other end of the wrinkled region are unknown. This is insufficient to provide the upper and lower limits of integral equation (3.5.8). Alternatively, for such problem, by continuity, the values of all geometrical variables are continuous at the interior boundaries, we can integrate the governing equations directly without resource to the integral form. We consider such deformations in section 3.6.

$F^d_c = 0.10$	$\lambda_o = 1.029776$	$t_b = 2.338007°$
$F^d_c = 0.15$	$\lambda_o = 1.044799$	$t_b = 3.827460°$
$F^d_c = 0.20$	$\lambda_o = 1.060647$	$t_b = 5.322790°$
$F^d_c = 0.25$	$\lambda_o = 1.07665$	$t_b = 7.467120°$
$F^d_c = 0.30$	$\lambda_o = 1.09340$	$t_b = 9.598969°$
$F^d_c = 0.35$	$\lambda_o = 1.11088$	$t_b = 13.146600°$
$F^d_c = 0.40$	$\lambda_o = 1.129357$	$t_b = 17.617758°$

Table 3.5.1 Solutions of λ_o at the equator. Tense zone t_b. $p^d = 0$.

$F^d_c = -0.01$	$\lambda_o = 1.159741$	$t_b = 84.31468°$
$F^d_c = -0.05$	$\lambda_o = 1.145966$	$t_b = 79.823255°$
$F^d_c = -0.10$	$\lambda_o = 1.129546$	$t_b = 72.278096°$
$F^d_c = -0.16$	$\lambda_o = 1.110593$	$t_b = 67.263036°$
$F^d_c = -0.18$	$\lambda_o = 1.104578$	$t_b = 45.947244°$
$F^d_c = -0.24$	$\lambda_o = 1.083753$	$t_b = 19.798545°$
$F^d_c = -0.26$	$\lambda_o = 1.073394$	$t_b = 4.5900000°$

Table 3.5.2 Solutions of λ_o at the equator. Tense zone t_b for internal pressure p_d =

1

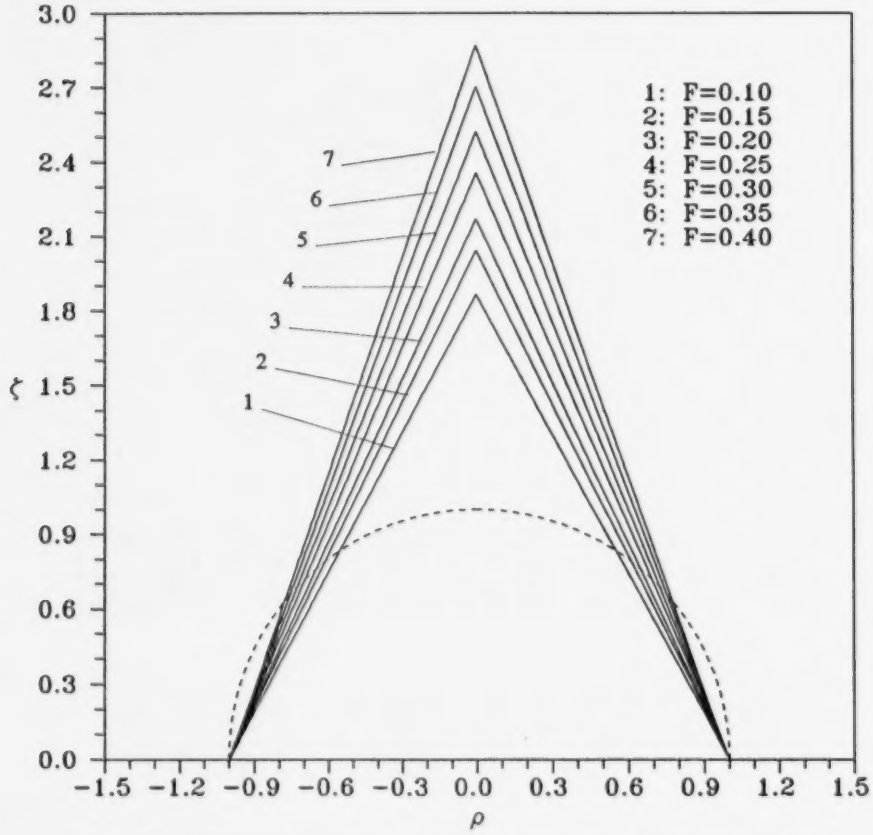

Fig.3.5.2 Profiles of the deformed meridians for point-load F alone

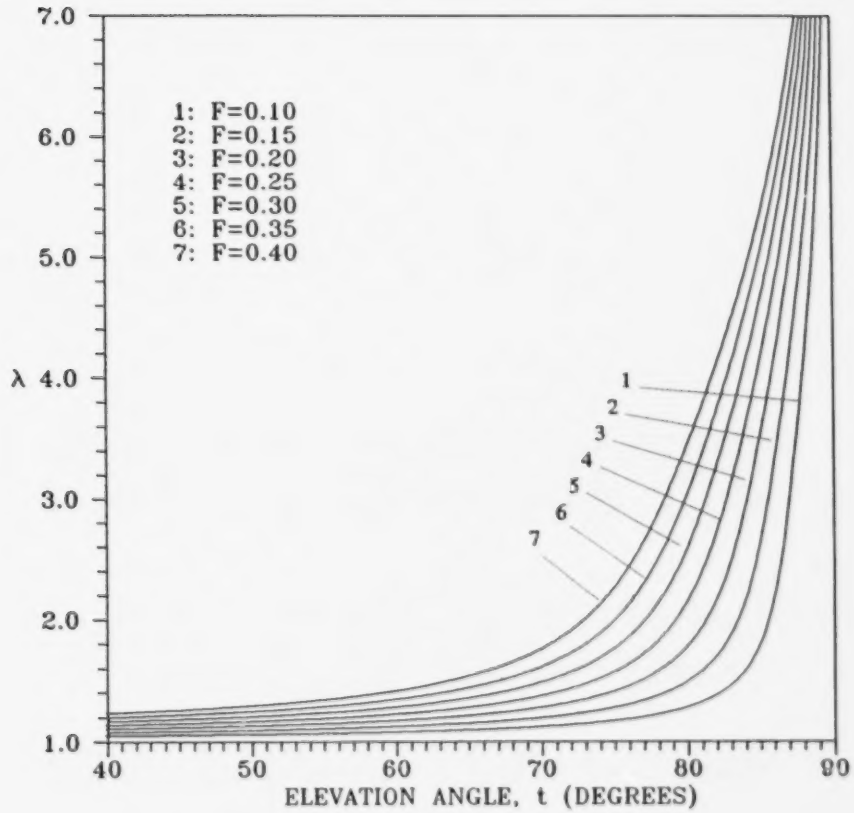

Fig. 3.5.3 Distribution of λ for point-load F alone.

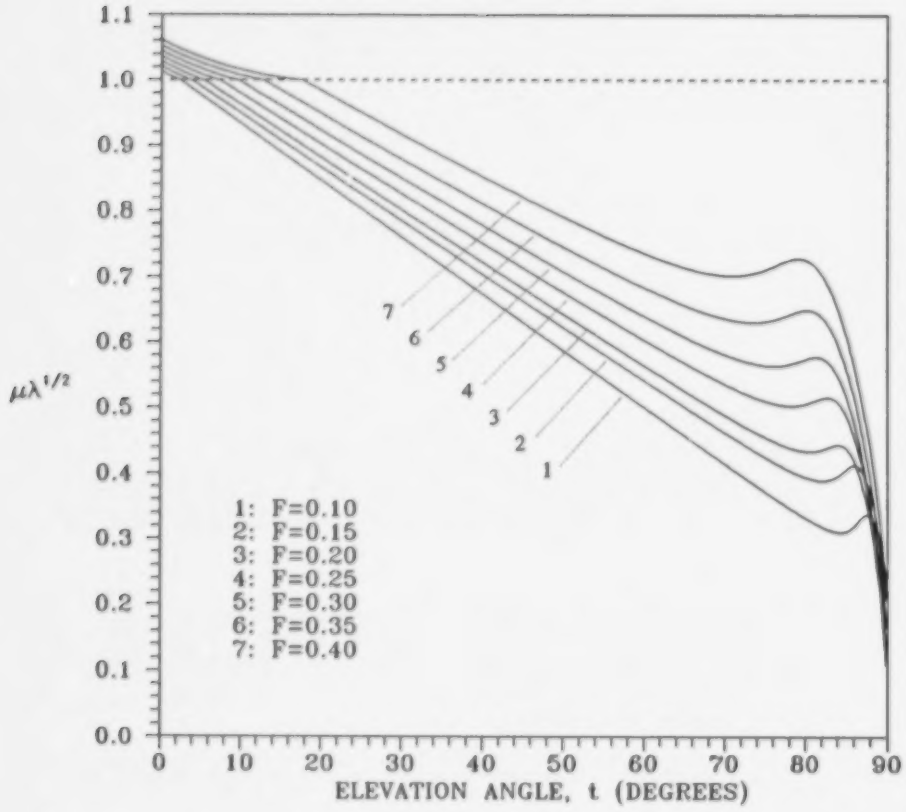

Fig. 3.5.4 Distribution of $\mu\lambda^{1/2}$ for point-load F alone. Interior boundary is denoted by $\mu\lambda^{1/2} = 1$. Above 1 is tense, whereas below 1 is wrinkled.

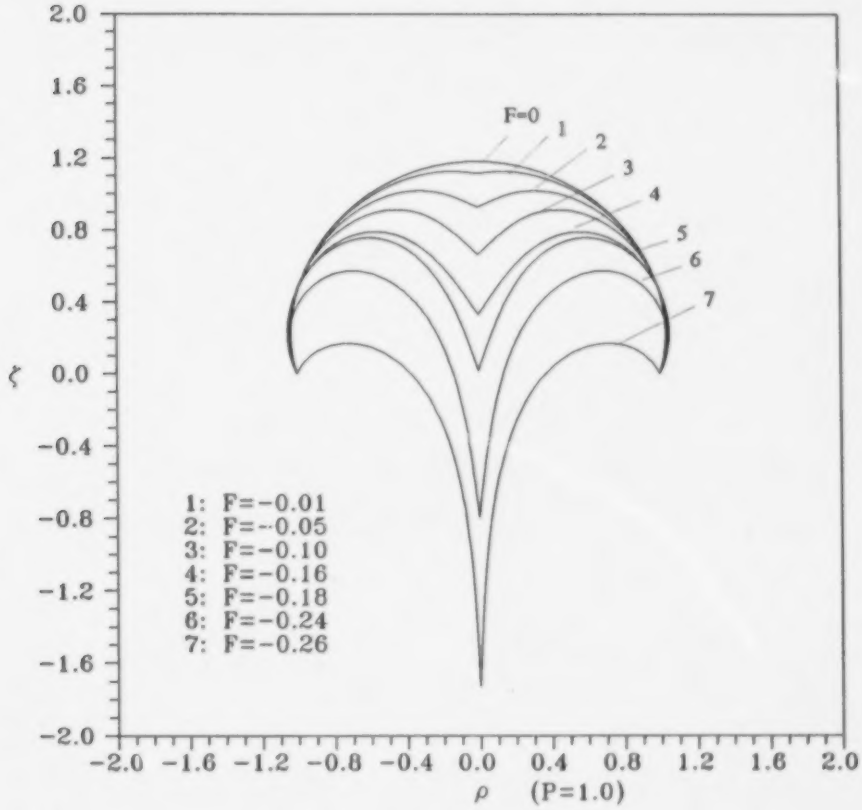

Fig. 3.5.5 Profiles of the deformed meridians for various point-loads F. (P=1)
F = 0 is the fully tense deformation for p=1.

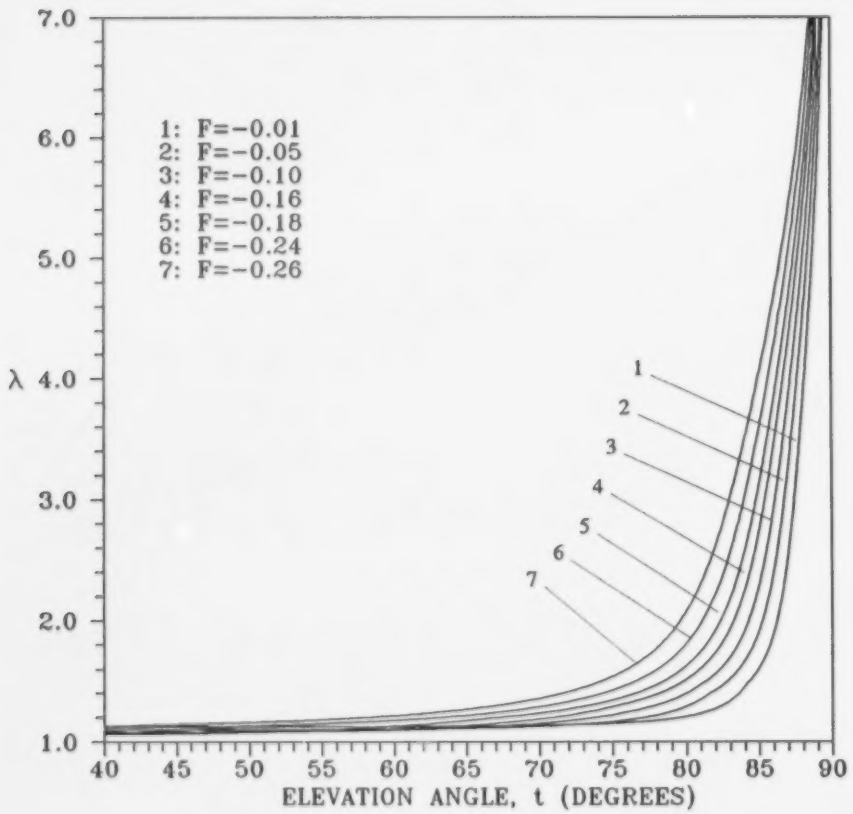

Fig. 3.5.6 Distribution of λ for various point-load values.

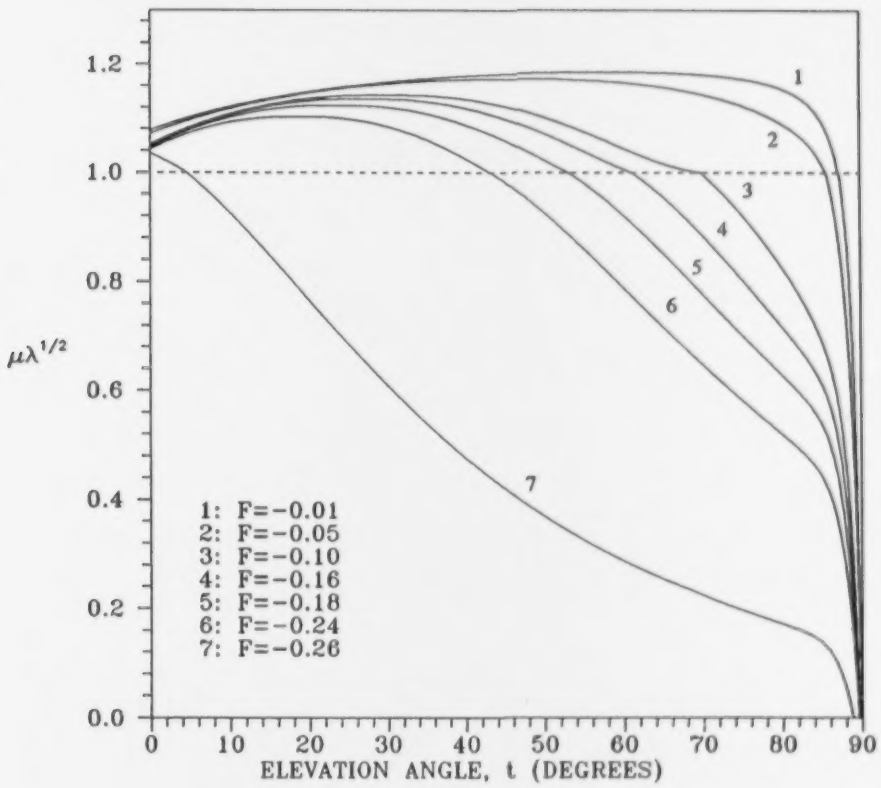

Fig. 3.5.7 Distribution of $\mu\lambda^{1/2}$ for various values of point-load F. Above $\mu\lambda^{1/2} = 1$, the membrane is tense. Below $\mu\lambda^{1/2} = 1$, the membrane is wrinkled.

3.6 Axisymmetric deformations of a Membrane Barrel

The second example which illustrates the application of tension field theory is to consider axisymmetric deformations of a hemispherical barrel. In this example, the reference configuration is a hemispherical surface cut off along a circle of latitude at a distance above the equator. In addition, a rigid disk of the same radius is attached to the circle of truncation. Fig.3.6.1 shows the diagram of a radial cross-section of the hemispherical barrel. Libai [3.8] analyzed a spherical membrane which is cut off parallely along circles of latitude at an equal distance above and below the equator (spherical barrel). The spherical barrel is then pulled apart by two equal and opposite axial forces on the rigid discs attached to the sphere along the circles of truncation. Libai's solutions indicate that the deformed surface is wrinkled in an interior region abutted by tense regions adjacent to the rigid discs. In this section, we use tension field theory to obtain a complete solution of deformation for a hemispherical barrel which is fixed at the equator and pulled apart by axial forces on the rigid disc.

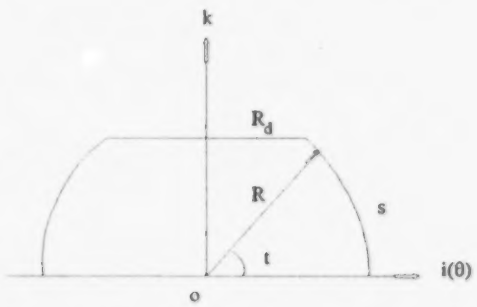

Fig.3.6.1 A truncated hemispherical membrane

For a barrel shown in Fig.3.6.1, the meridional curve can be described parametrically by equation (3.3.1). The location of the circle of truncation can be determined by the relation

$$t_d = \pi/2 - \sin^{-1}(R_d/R) \tag{3.6.1}$$

where R_d is the radius of the disc attached at the circle of truncation. It is assumed that $R_d < R$.

We consider the deformations caused by an axial force $\underline{F}^d = F^d_{\bullet}\,\underline{k}$ ($F^d_{\bullet} > 0$) applied upwardly-directed at the enter of the top disc. $F^d_{\bullet}$ is the prescribed magnitude of the force. By symmetry, the disc moves axially along $\underline{k}$. According to the non-dimensionalization scheme (3.3.5), the boundary conditions are

$$\mu(0) = 1, \quad \zeta(0) = 0 \quad \text{and} \quad \mu(t_d) = 1, \tag{3.6.2}$$

where t_d is given by (3.6.1).

Guided by Libai's solution [3.14], we assume that the deformed membrane is wrinkled in an interior region as the result of the pulling at the disc. This assumption will be verified a posteriori.

The complete derivation of the governing equations were given in section 3.3 where a set of equilibrium equations were presented in (3.3.14). In the wrinkled region, the strain energy $w(\lambda,\mu)$ must be replaced by the corresponding relaxed strain energy $w^R(\lambda)$. Considering the fact that $w^R_{\mu} = w^R_{\mu\lambda} = 0$ and setting $p^d = 0$ in $(3.3.14)_{1,4}$ for the absence of internal pressure, the governing equations take the form

$$\begin{aligned} \dot{\lambda} &= -\, w^R_{\lambda}\dot{r}/(r w^R_{\lambda\lambda}) \\ \sin\phi &= F^d / \cos(t)\, w^R_{\lambda}\,. \end{aligned} \tag{3.6.3}$$

Equations (3.6.3) and $(3.3.14)_{2,3}$ constitute the governing equations for the wrinkled region, while original equations (3.3.14) are the governing equations for the tense region.

In section 3.5, we used the properties of relaxed strain energy to justify our hypothesis on the tense region adjacent to the rigid equator. Similarly, we apply the same reasoning

here to the regions adjacent to the fixed equator and the rigid disc, respectively, it is concluded that the deformed membrane is tense in two end-regions $[0, t_b^1]$ and $[t_b^2, t_d]$ for some $t_b^1 > 0$, $t_b^1 < t_b^2 < t_b$. t_b^1 and t_b^2 are the interior boundaries separating the tense and wrinkled regions, the locations of which are not known a priori and the determination of them constitute a part of the problem.

The approach to the solution is based on the fact that a sufficient number of known conditions exist at the equator $t = 0$ and the boundaries t_b^1, t_b^2. This suffices to transform the boundary value problem to an initial value problem.

Numerical examples are presented for Ogden's strain energy function (3.1.3). To solve the problem, we prescribe a value of F^d and guess the value λ_o (> 1) of λ at the equator. In addition, $\mu_o = 1$ comes from boundary condition (3.6.2). Using these initial values, we calculate angle ϕ at $t = 0$ from $(3.6.3)_2$. These data suffice to initiate the numerical integration of equations (3.3.14) using a 4^{th} order forward-marching Runge-Kutta scheme from $t = 0$ to $t = \Delta t$, where Δt is a suitable step size. At $t = \Delta t$, ϕ is calculated using $(3.6.3)_2$ upon the available values of λ, μ and $r(\Delta t)$. The values of λ, μ, ϕ and ζ are then updated and the procedure is repeated until the first boundary point t_b^1 is reached at which λ_b^1, μ_b^1 calculated satisfy the condition $\mu_b^1 = \upsilon(\lambda_b^1)$. υ is the natural width defined in (3.1.8). Also at t_b^1, quantities $\rho_b^1 = \rho(t_b^1)$, $\zeta_b^1 = \zeta(t_b^1)$ and $\lambda_b^1 = \lambda(t_b^1)$ are calculated. Continuity and equilibrium between the tense and wrinkled parts require that λ, μ, ρ, ξ and ϕ be continuous at t_b^1. Assuming the membrane to be wrinkled for $t > t_b^1$, one may use λ_b^1, μ_b^1, ϕ_b^1 and ζ_b^1 obtained from the tense region as the initial values to integrate equations (3.6.3) and $(3.3.14)_{2,3}$ to update the corresponding values at the successive point. This procedure is terminated at the second boundary point t_b^2 at which

$\mu_b^2 = \upsilon(\lambda_b^2)$ is satisfied. The ending values λ_b^2, μ_b^2, ϕ_b^2 and ζ_b^2 are in turn used as the initial values for integrations in the remaining tense region $t \in (t_b^2, t_d]$. Using λ_b^2, μ_b^2, ϕ_b^2 and ζ_b^2 as initial guesses, we then go back to (3.3.14) to integrate these equations until t_d is encountered. At t_d, values of μ_d is recorded. By varying λ_o, μ_d may be plotted against λ_o and the solutions of λ_o is found such that μ_d satisfies the boundary condition $(3.6.2)_3$. For each prescribed F^d, values of λ, μ, ϕ and ζ at t_d computed are uniquely determined by λ_o. Solutions of λ_o for various values of F^d are tabulated in Table 3.6.1.

Graphical examples are presented for a hemisphere with unit radius truncated at $t_d = \pi/3$ and attached to a disk with $R_d = 0.5$. Fig.3.6.2 and Fig.3.6.3 show the deformed meridians and the distribution of λ for $F^d = 0.1, 0.2, 0.3, 0.4$. To verify the early hypothesis that the membrane is tense adjacent to the equator and rigid disc and wrinkled in the interior region, we plot stretch $\mu(t)$ and $\upsilon(\lambda)$ in Fig.3.6.4. To avoid crowding, only this solution for $F^d = 0.4$ is plotted. From the figure it is evident that $\mu > \upsilon(\lambda)$ in tense regions, whereas $\mu < \upsilon(\lambda)$ in the wrinkled region. This is in accordance with the definition of the relaxed strain energy.

We have also verified that the solutions obtained satisfy the Legendre-Hadamard conditions at all mesh points in tense regions $t \in [0, t_b^1]$ and $t \in (t_b^2, t_d]$. The conditions are automatically satisfied in the wrinkled region $t \in (t_b^1, t_b^2)$. Since λ and μ are continuous and finite in the interval $t \in (0, t_d)$ and at the boundary points $t = 0$ and t_d, the strain energy function $w(\lambda,\mu)$ and the relaxed strain energy function $w^R(\lambda)$ are also finite. Therefore, the total potential energy (3.3.3) for $p^d = 0$ is also finite. This ensures the applicability of minimum energy principle.

Once again, Fig.3.6.3 shows finite jump discontinuities in $\dot{\lambda}$ and $\dot{\mu}$ at t_b^1 and t_b^2. Numerical values of the finite jumps may be calculated based on the formula (3.5.22).

$F^d = 0.1$	$\lambda_o = 1.0146$	$t_b^1 = 1.0799°$	$t_b^2 = 58.6199°$
$F^d = 0.2$	$\lambda_o = 1.0295$	$t_b^1 = 2.4000°$	$t_b^2 = 56.9800°$
$F^d = 0.3$	$\lambda_o = 1.0450$	$t_b^1 = 3.7150°$	$t_b^2 = 55.3796°$
$F^d = 0.4$	$\lambda_o = 1.0612$	$t_b^1 = 5.0999°$	$t_b^2 = 53.6998°$

Table 3.6.1 Solutions of λ_o at the equator for various values of force F^d on the rigid disc. t_b^1 is the tense zone near the equator. t_b^2 is the tense zone near the rigid disc. (t is the elevation angle measured from the $\underline{i}(\theta)$ axis)

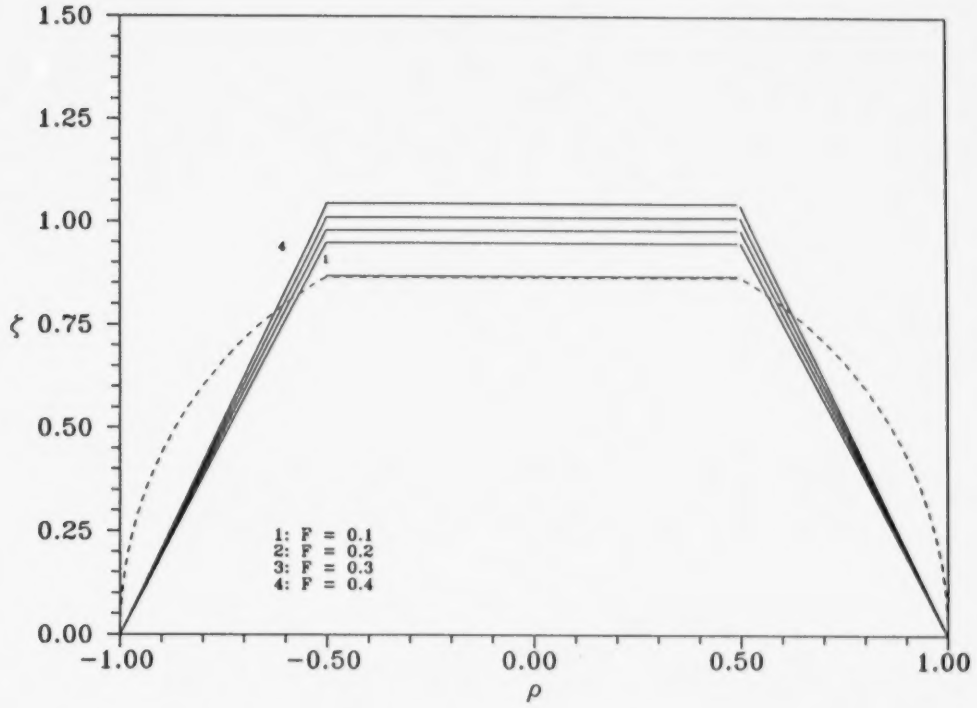

Fig. 3.6.2 Profiles of the deformed meridians for various values of F^d.

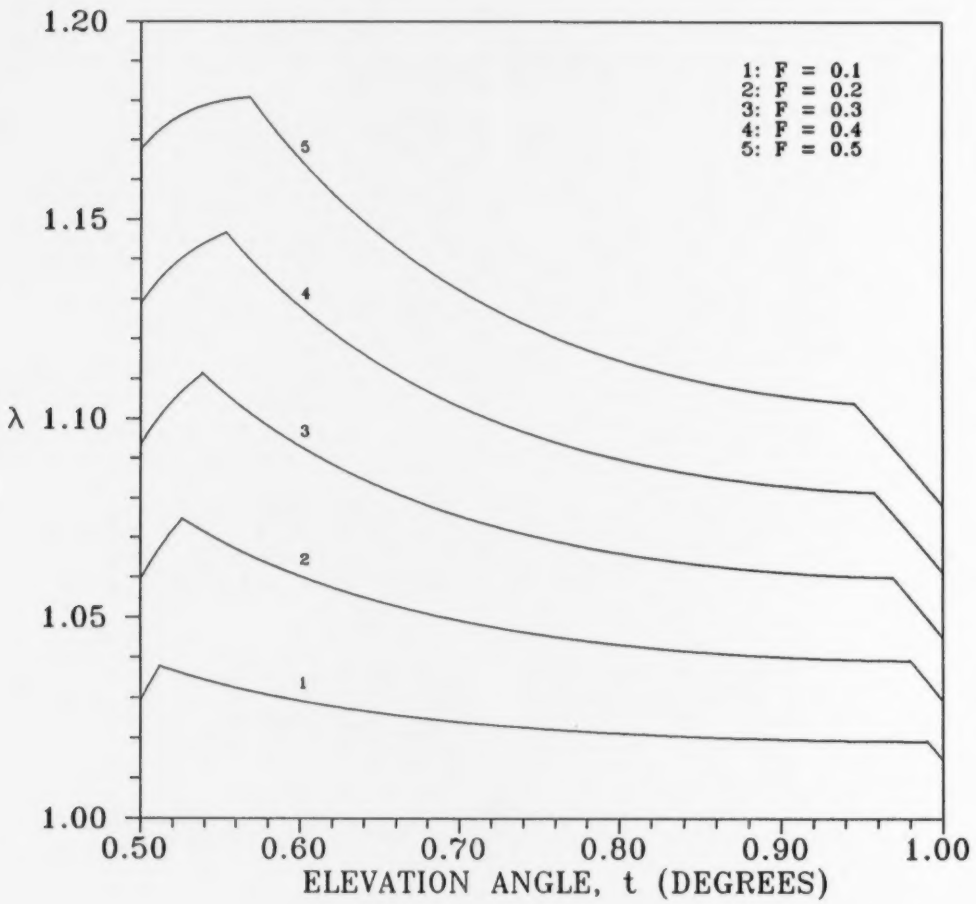

Fig.3.6.3 Distribution of λ for various values of F^d.

Cusp points indicate the interior boundaries. The membrane is wrinkled when $\mu < v(\lambda)$, while membrane is tense when $\mu > v(\lambda)$.

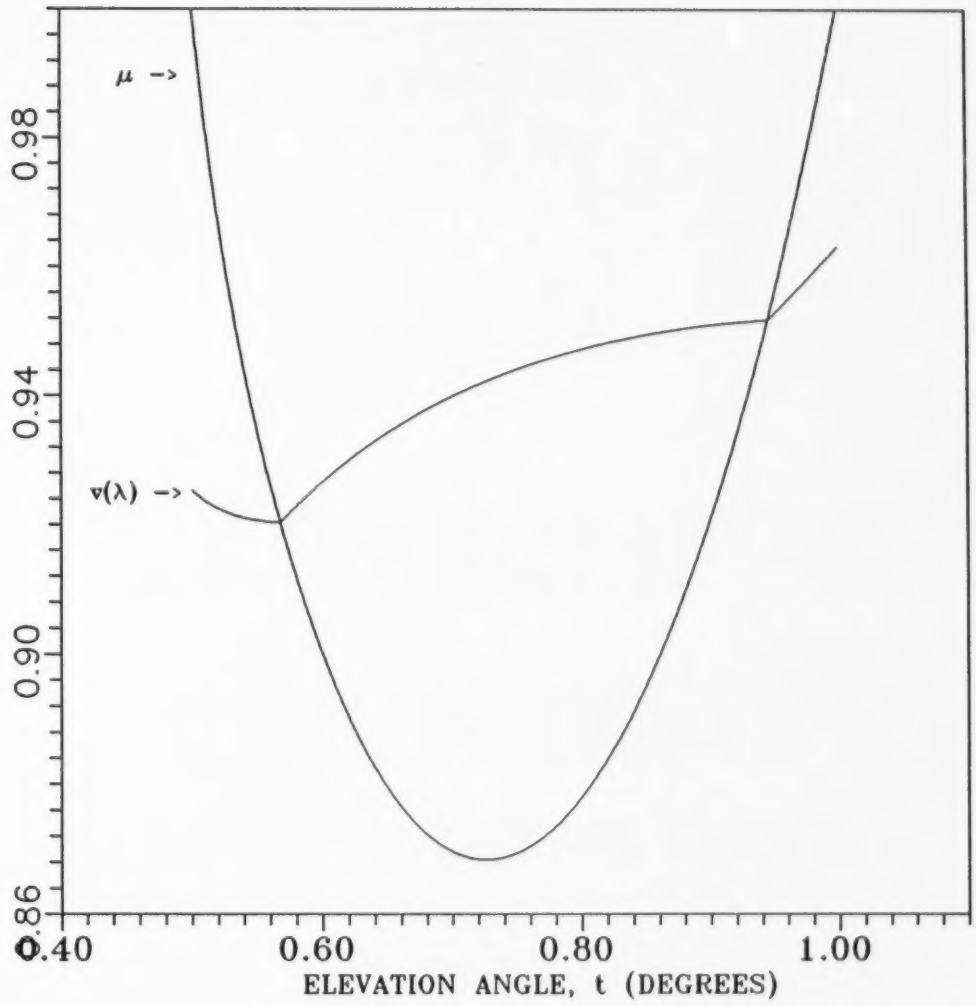

Fig.3.6.4 Distribution of μ and the natural width $\upsilon(\lambda)$ for $F^d = 0.4$.

3.7 A closed circular sheet pulled apart by opposing axial point forces

In this section, we consider a closed surface of revolution subjected to equal and opposite axial point loads at its apices. In the presence of point-loads alone, Steigmann [3.1] has shown that for any amount of axial point-load the initially unstrained membrane will collapse onto the axis of symmetry. He also demonstrated that the deformed membrane surface is entirely wrinkled. In this section, using the equation $rw^{R}_{\lambda}(\cdot) = c$ derived in section 3.5, we obtained a solution for a closed circular membrane which is flat in its initial state and is pulled apart axially at its apices.

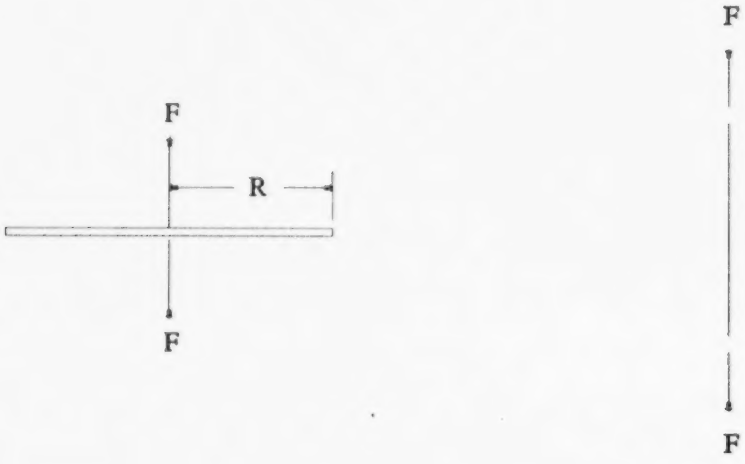

Fig. 3.7.1 Reference and deformed configurations of a closed circular membrane.

As shown in Fig.3.7.1_a, the reference surface is a composition of two plane circular membrane sheets occupying the region $0 \le r \le R$ in planes $z = 0^-, 0^+$, respectively. Two membrane sheets are joined along the boundary $r = R$. In view of (3.2.1), the position of the point on the reference membrane is determined by the vector

$$\underline{x}(r, \theta) = r\,\underline{i}(\theta), \qquad r \ge 0,\ z = 0^-, 0^+ \tag{3.7.1}$$

where r is the radial coordinate.

Suppose that the membrane is pulled apart by two equal and opposite point-loads F^d_c at its poles, consequently, the membrane collapses onto the $\underline{k}$-axis axisymmetrically. The deformed configuration is shown in Fig. $3.7.1_b$. By symmetry, the points along the outer boundary ($r = R$) move only radially in the plane $z = 0$ during the deformation. On the other hand, the point at the pole moves only along the $\underline{k}$-axis. Since the deformation is also symmetric about plane $z = 0$, it is sufficient to only consider the deformation in the plane $z = 0^+$. The deformation in the plane $z = 0^-$ is a mirror reflection of that in the plane $z = 0^+$.

According Fig. $3.7.1_b$, the deformation is

$$\underline{r}(r, \theta) = \xi(r)\, \underline{k};\ \ \xi \geq 0,\ 0 \leq \theta \leq 2\pi \tag{3.7.2}$$

in which an initial horizontal straight-line meridian is mapped onto the vertical $\underline{k}$-axis.

Nondimensional quantities are introduced in (3.3.5) with respect to the reference radius R and the material constant G associated with the relaxed strain energy function $w^R(\lambda)$ defined by (3.1.9). To accommodate current variables, s is replaced by r. The non-dimensional reference configuration is a circular disc of unit radius.

It is easy to specify the boundary conditions which are

$$\rho(1) = 0,\ \ \rho(0) = 0,\ \ \xi(1) = 0 \tag{3.7.3}$$

Since the deformation (3.7.2) is axisymmetric, kinematic relations derived in section 3.2 for the surfaces of revolution are readily applicable with slight notational difference. In this case, relation (3.2.10) becomes

$$\lambda = -\,\xi'(r),\ \ \mu = 0\ (r \neq 0) \tag{3.7.4}$$

where the minus sign is chosen to accommodate the direction convention dr = - ds. The prime denotes the differentiation with respect to r.

In view of equations (3.2.11), it is easy to show that the principal vectors are $\underline{l} = \underline{k}$, $\underline{m} = \underline{0}$. Same conclusion may also be deduced from $(3.2.13)_1$, which implies that

$$\phi = \pi/2. \tag{3.7.5}$$

We assume that the entire upper membrane $z = 0^+$ is wrinkled due to the application of $\underline{F}^d_c = F^d_c\underline{k}$ at the pole. Based on the wrinkling hypothesis, the total potential energy of the deformation can be written as

$$E/2G\pi = \int_0^1 r\, w^R(\lambda)\, dr - F^d_c \xi(0) \tag{3.7.6}$$

A single Euler-lagrangean equation for the functional (3.7.6) can be derived in a straightforward manner. Considering boundary conditions (3.7.3), the integral form of the equation can be presented in the form

$$r \cdot w^R_\lambda(\lambda) = F^d_c \tag{3.7.7}$$

in which relation $(3.7.4)_1$ is used. (3.7.7) appears identical in form compared with equation (3.5.1). This relation determines a uniaxial stress distribution along the initial meridian z = 0.

We use equation (3.7.7) to transform the independent variable from r to λ on a one-to-one basis. The advantage of using λ, instead of r, as the independent variable was evidently demonstrated in section 3.6 in solving the problem with pole-type singularities in the governing equations.

To proceed with the transformation of independent variable, we first differentiate (3.7.7) which yields

$$dr = -r \frac{w^R_{\lambda\lambda}}{w^R_{\lambda}} d\lambda \tag{3.7.8}$$

From $(1.3.10)_{1,3}$, we have $w^R_{\lambda}(\cdot) > 0$, $w^R_{\lambda\lambda}(\cdot) > 0$ in the domain $[1, \infty)$. Therefore, derivative $dr/d\lambda$ is a monotonically decreasing function with increasing λ. This ensures the one-to-one correspondence between r and $\lambda \in [1, \infty)$.

Substituting relation (3.7.8) into equation $(3.7.4)_1$ and use (3.7.7) to eliminate r, we arrive at the equation

$$d\xi = F^d_c \frac{w^R_{\lambda\lambda}}{(w^R_{\lambda})^2} d\lambda \tag{3.7.9}$$

Since F^d_c is a constant, the stress resultant $w^R_{\lambda}(\lambda) \to \infty$ as $r \to 0$. As a result, $\lambda \to \infty$ as $r \to 0$ for the relaxed strain energy (3.1.9). If λ_1 is used to denote the value of λ at the outer boundary $r = 1$, ∞ denotes the limiting value of λ as $r \to 0$. For $F^d_c > 0$, it is seen that $d\xi/d\lambda > 0$ in the domain $[\lambda_1, \infty)$. Moreover, from (3.1.10) and (3.1.11), the leading terms in $w^R_{\lambda}(\cdot)$ and $w^R_{\lambda\lambda}(\cdot)$ are λ^{p-1} and λ^{p-2} $(p > 1)$, respectively, as $\lambda \to \infty$. In view of (3.7.8), $d\xi/d\lambda$ is shown to decay at the rate of $1/\lambda^p$ as $\lambda \to \infty$. Therefore, the vertical displacement of the pole is finite.

Considering boundary conditions (3.7.3), the relation between the axial deformation component ξ and λ is determined by integrating (3.7.9), which yields an integral equation

$$\xi(\lambda) = \left[\frac{\lambda\, F^d_c}{w^R_{\lambda}(\lambda)}\right]^{\lambda}_{\lambda_1} + \int_{\lambda_1}^{\lambda} \frac{F^d_c}{w^R_{\lambda}(\lambda)} d\lambda \tag{3.7.10}$$

where λ_1 and λ are the roots of equation (3.7.7) at $r = 1$ and an arbitrary interior point $r\ (<1)$, respectively. Thus ξ is an implicit function of r.

To find the relation between F^d_c and the deflection ξ at the pole (r=0), we first prescribe a value for F^d_c and solve for λ_1 (> 1) from the equation (3.7.7) to determine λ at the outer boundary $r = 1$. The deflection of the pole (r=0) is then calculated from equation (3.7.10) by setting the upper and low limits to ∞ and λ_1. To evaluate the integral, we make the change of variable $\lambda \to x^{-1}$ and integrate from $x = 1/\lambda_1$ to $x = 0$. Numerical examples for $F^d_c = 0.05, 0.1, 0.15, 0.2$ are presented in Table 3.7.1.

To verify that the values of total potential energy (3.7.6) are finite for the solutions obtained, we use (3.7.8) and (3.7.9) to write the potential energy (3.7.6) in the form

$$E/2\pi = [F^d_c]^2 \int_{\lambda_1}^{\infty} \left[\frac{W^R}{W^R_\lambda} - 1\right] \left[\frac{W^R_{\lambda\lambda}}{W^R_\lambda}\right] d\lambda \qquad (3.7.11)$$

We use the relaxed strain energy function of Ogden's material defined by (3.1.9), the derivatives $w^R_\lambda(\cdot)$ and $w^R_{\lambda\lambda}(\cdot)$ are derived in (3.1.10) and (3.1.11). It is shown that $w^R(\cdot)$, $w^R_\lambda(\cdot)$ and $w^R_{\lambda\lambda}(\cdot)$ behave like λ^p for large λ. Evidently, the leading term of the first term in the first square bracket of the integrant (3.7.11) behaves asymptotically like λ as $\lambda \to \infty$, whereas the term in the second square bracket tends asymptotically to be of function of λ^{-5} for very large λ. Thus the resultant asymptotic function becomes λ^{-4} which decays to zero more rapidly than λ^{-1} as $\lambda \to \infty$. Therefore, existence of a finite energy is once again secured.

$F^d_c = 0.05$	$\lambda_1 = 1.0169970$	$\xi_o = 3.2576270$
$F^d_c = 0.10$	$\lambda_1 = 1.0346930$	$\xi_o = 5.7498559$
$F^d_c = 0.15$	$\lambda_1 = 1.0531340$	$\xi_o = 7.9091864$
$F^d_c = 0.20$	$\lambda_1 = 1.0723550$	$\xi_o = 9.8314404$

Table 3.7.1 Values of λ_1 at the outer boundary for various values of F^d_c.

ξ_o denotes the axial deformation of the apex.

3.8 A cylindrical membrane tube to a circular plane

In this section we study the problem of deforming a cylindrical tube into a flat circular plate. Reference cylindrical tube is shown in Fig.3.8.1

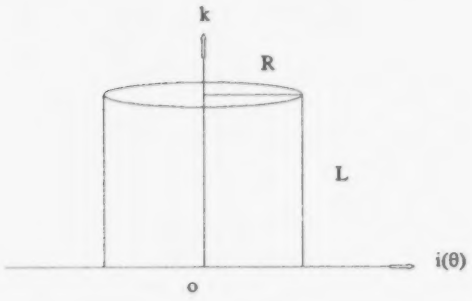

Fig. 3.8.1 A cylindrical tube

The deformation is performed successively in the following way. We first apply a radial stretch along the bottom end ($z = 0$) to stretch the tube into a flat annulus in the plane $z = 0$. In this state, the original upper end ($z = L$) forms the inner boundary of the annulus. This deformation was first described by Wu [3.10]. Superposing on Wu's solution, we then constrict the inner boundary to the centre of the symmetry, namely point o, to form a circular plane. The deformed surface is wrinkled in the central area. The deformed configuration is shown in Fig.3.8.2 where dashed lines indicate the wrinkled area.

The solution procedure used is a semi-inverse method in the sense that we specify the radial displacement at the bottom end and find the stretch required at the upper end.

Therein, the deformed and undeformed geometries of the body are prescribed and the forces necessary to maintain the deformed body in a state of equilibrium are calculated.

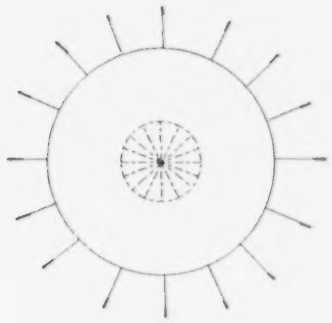

Fig.3.8.2 Circular plate. Dashed area is wrinkled

For the reference tube as shown in Fig.3.8.1, we choose a cylindrical polar coordinate system (r,θ,z), the z-axis of which coincides with the axis of symmetry of the tube. Hence, the reference tube can be described by the position vector (3.2.1) with components

$$r(z) = R, \qquad 0 \le z \le L,\ 0 \le \theta \le 2\pi. \tag{3.8.1}$$

where s is the arclength measured along the meridian form $z = 0$.

A point with coordinates (R,θ,z) on the tube is displaced axisymmetrically to the new position $(\rho,\theta,0)$ in the plane $z = 0$, the deformation can be described by

$$\underline{r}(z,\theta) = \rho(z)\ \underline{i}(\theta), \qquad \rho \ge 0,\ 0 \le z \le L,\ 0 \le \theta \le 2\pi. \tag{3.8.2}$$

It is natural to work with dimensionless variables. We adopt the nondimensionalization scheme (3.3.5) with respect to the reference radius and the material constant G associated

with the strain energy function. By axisymmetry of the deformation, radial stretch λ and circumferential stretch μ are determined by (3.2.10) which, for the deformation (3.8.2), take the form

$$\lambda = \dot{\rho}(z) \quad \text{and} \quad \mu = \rho(z) \tag{3.8.3}$$

where the super-dot denotes the differentiation with respect to the nondimensional z.

The appropriate boundary conditions are given as

$$\xi(0) = 0, \quad \rho(L) = 0, \quad \xi(L) = 0 \tag{3.8.4}$$

In view of (3.8.3), the total potential energy can be written as an integral

$$\frac{E}{2\pi RG} = \int_0^L w(\rho, \rho')dz \tag{3.8.5}$$

where L is the nondimensional tube height. Considering boundary condition (3.8.4), the single equilibrium equation is

$$(w_\lambda)^{\cdot} = w_\mu. \tag{3.8.6}$$

From (3.8.3), the compatibility equation is

$$\dot{\mu} = \lambda \tag{3.8.7}$$

Using the fact that strain energy function $w(\cdot)$ for homogeneous materials is not explicitly dependent on z, equilibrium equation (3.8.6) can be easily cast into the form

$$\dot{\lambda} = (w_\mu - \lambda\, w_{\lambda\mu}) / w_{\lambda\lambda} \tag{3.8.8}$$

provided $w_{\lambda\lambda} \neq 0$. Here, equation (3.8.7) has been used.

Equations (3.8.7), (3.8.8) constitute a system of differential equations of the first order in λ and μ. Once the boundary conditions at $z = L$ are appropriately prescribed, these equations can be integrated as an initial value problem in the interval $z \in [0, L]$.

For the deformation (3.8.2), wrinkling emerges in the central area on the deformed circular membrane. This particular behaviour may be explained by the following reasoning. From $(3.8.4)_2$ and (3.8.3), it implies

$$\mu \to 0 \text{ as } z \to L \tag{3.8.9}$$

Due to the inwardly radial stretch of the upper end, in the neighbourhood of $z = L$, natural width $v(\lambda) = \lambda^{-1/2} > 0$ due to $\lambda > 1$. This indicates that $\mu < v(\lambda)$ in a region from $z = L$ up to a point z_b at which condition $\mu_b = v(\lambda_b)$ is satisfied. Furthermore, the deformation is partly caused by the outward stretch at the bottom end. Thus, according to (3.8.3), the deformed radius μ must be greater than $v(\lambda)$ in the remaining interval $z \in [0, z_b]$. Based on this analysis, we assume that the membrane is tense in the interval $[0, z_b]$ and wrinkled in the remaining interval $[z_b, L]$. In the wrinkled region, relaxed strain energy of Ogden's material defined by (3.1.9) is used. In the region $[z_b, L]$, the membrane is tense, the ordinary strain energy function (3.1.3) is used. Wrinkling assumption is verified *a posteriori* (see Fig.3.8.2).

Relaxed strain energy function is inherently independent of the circumferential stretch μ. Consequently, we have $w^R{}_\mu(\lambda) = 0$, $w^R{}_{\lambda\mu}(\lambda) = 0$ in the interval $[z_b, L]$. Hence, use of $w^R(\lambda)$ in (3.8.8) results in that λ is a constant in the interval $[z_b, L]$. It is

$$\lambda = \lambda_w. \tag{3.8.10}$$

where λ_w (>1) is a constant to be determined. As a result, compatibility equation (3.8.8) yields that μ is a linear function of z in the interval $[z_b, L]$

$$\mu = \lambda_w z \tag{3.8.11}$$

provided that λ_w is known.

Using equation (3.8.11), the wrinkled region can be determined in terms of λ_w. At z_b, condition $\upsilon(\lambda_w) = \lambda_w^{1/2}$ must be satisfied. Substituting (3.8.11) into (3.1.8), the radius of the wrinkled area is obtained as

$$z_b = \lambda_w^{-3/2} \tag{3.8.12}$$

To solve the particular problems, we specify λ_w and use this value to determine the wrinkled region z_b by (3.8.12). The value of μ at z_b, denoted by μ_w, is related to λ_w by satisfying the condition $\mu_w = \upsilon(\lambda_w)$ at z_b. Thus, only one parameter needs to be specified. Using λ_w, μ_w, we integrate equations (3.8.7) and (3.8.8) by a forward marching Runge-Kutta method from $z = z_b$ to a successive point $z = z_b + \Delta z$ at which the values of λ and μ are updated. Δz is a suitable step size for numerical integration. The preceding integration procedure is terminated at $z = L$. Numerical examples are presented for five values of λ_w, specifically, $\lambda_w = 2.0, 2.5, 3.0, 3.5, 4.0$.

The circumferential principal stretch μ $(= \rho)$ is plotted against z in Fig.3.8.3. It is seen that $\mu < \upsilon(\lambda)$ in the interval $[0, z_b]$ and $\mu > \upsilon(\lambda)$ for $z > z_b$. This verifies our early hypothesis. In Fig.3.8.3, it is interesting to note that values of natural width $\upsilon(\lambda)$ approach an asymptote (=0.495) very rapidly for values of λ_w we have prescribed. Also Fig.3.8.3 shows that μ $(= \rho)$ is a monotonically increasing function of z. In Fig.3.8.4, λ is shown to have a minimum near z_b, we denote this minimum and its location by λ_m and z_m, respectively. This result is in qualitative agreement with those obtained by Wu [3.10] in which the deformed membrane is fully tense, however. Fig.3.8.4 also shows the fact that z_m moves towards z_b with the increasing λ_w while the difference $(\lambda_w - \lambda_m)$ decreases as λ_w increases. We infer that for λ_w large than a certain value, z_m disappears and λ becomes a monotonically function of z in the tense region.

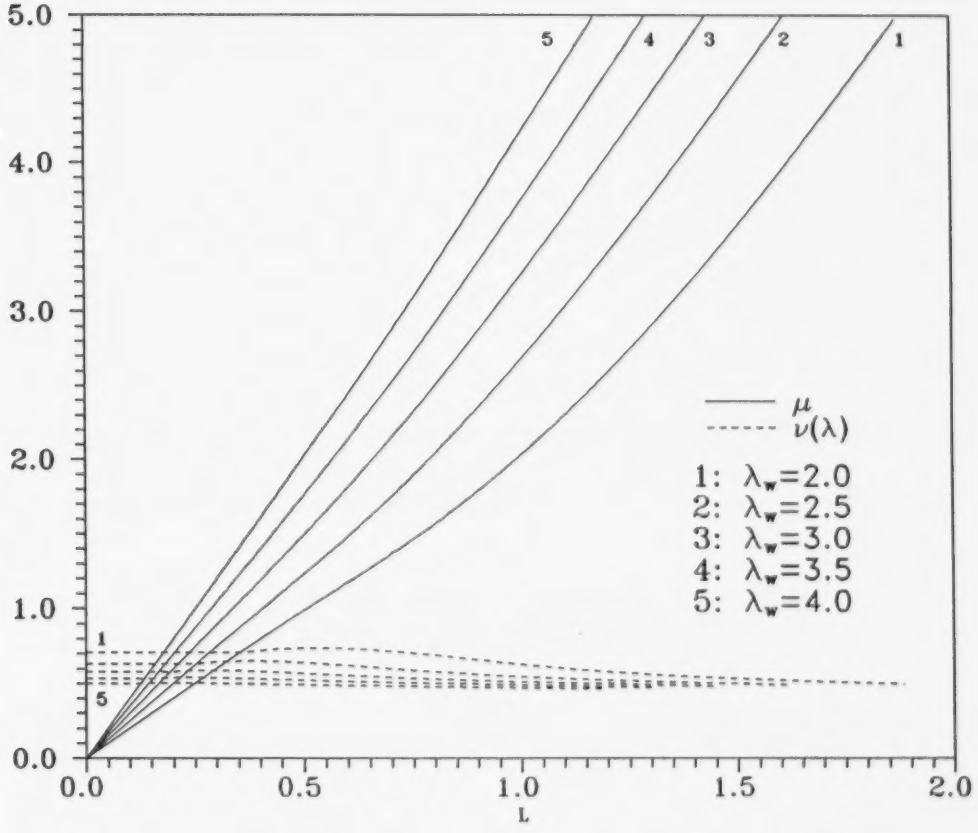

Fig.3.8.3 Distributions of μ and υ along the reference meridian.

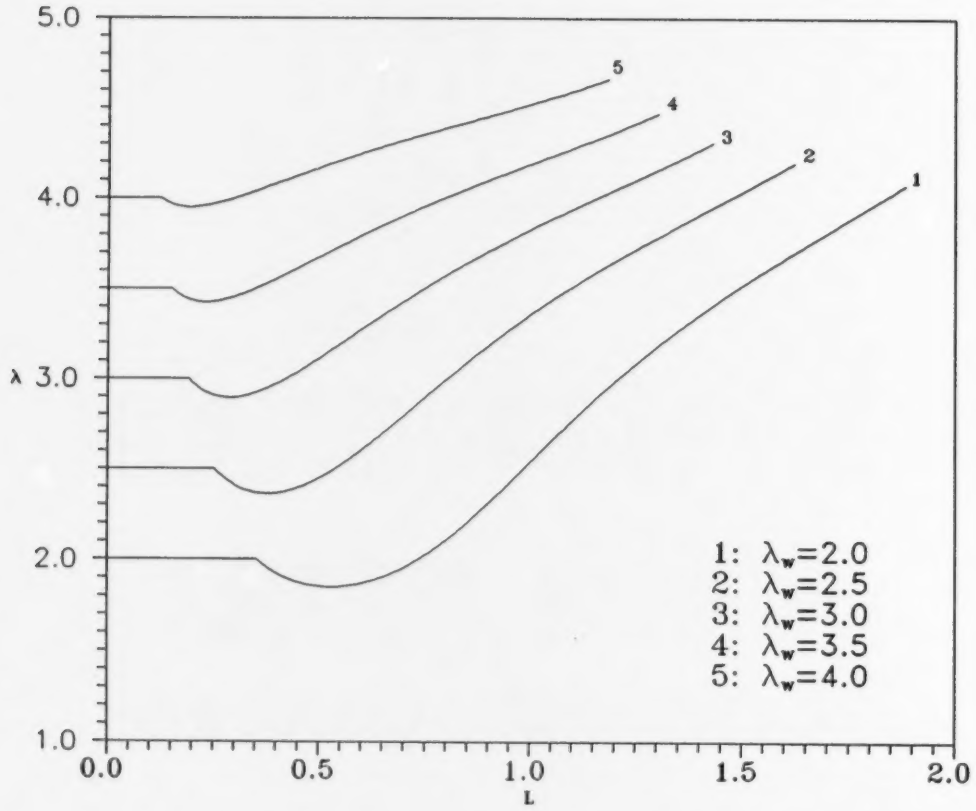

Fig. 3.8.4 Distribution of λ along the reference meridian.

Section 3.9 Axisymmetric deformations of a tubular membrane torus

The first satisfactory treatment of the nonlinear closed toroidal membrane was carried out by Kydoniefs [3.12]. Using the membrane theory developed by Green and Adkins [1.14], he considered the finite inflation of a toroidal membrane of the circular cross section. Differential equations were solved by using a perturbation series in powers of the ratio of the radius of the circle generating the torus to the overall radius of the torus. A number of useful numerical solutions were given for the Mooney strain energy function. Another work by Kydoniefs [3.13] considered the axisymmetric deformation of an elastic membrane torus whose axial meridian is symmetric about the horizontal plane and otherwise arbitrary. The solutions were sought following the same manner as that in [3.12]

Our main interest is the nonlinear treatment of a closed toroidal membrane with a smooth but otherwise arbitrary axial cross-section. In particular, we consider the membrane torus which exhibits wrinkles when it is finitely inflated by the internal pressure.

As shown later, solvability of equilibrium equations are significantly influenced by the reference geometry. In general, for the arbitrarily given reference configuration, both analytic and numerical solutions to the equilibrium equations do not seem feasible. However, it is possible to obtain some solutions if reference geometry possesses certain symmetry.

As an example, we consider a tubular torus which is formed by two concentric membrane tubes of the same height whose radii are infinitesimally close. They are

joined along the end boundaries. The torus is then inflated by the internal pressure. The torus is partly wrinkled in the inner region for a certain range of the pressure. Moreover, the results show that the size of wrinkled areas is markedly influenced by the height/radius ratio of the tube for a given range of the prescribed pressure. This is a problem of intrinsic interest in regard to Libai's conjecture [3.6] about wrinkling of a membrane torus which possesses some peculiar geometrical characteristics in its undeformed configuration. We do not attempt to give a general conclusion but only seek solutions of a particular problem.

For convenience, a brief summary of the general equations pertaining to finite axisymmetric deformations of the toroidal membrane is given in section 3.9.1 using nonlinear elastic membrane theory. These equations are presented in a non-dimensional form and no restriction is imposed on the form of strain energy function at this stage.

3.9.1 Axisymmetric deformations of a toroidal membrane.

We consider a membrane which in its undeformed state is a torus referred to an orthogonal cylindrical polar system $\{\underline{e}_r(\theta), \underline{e}_\theta(\theta), \underline{k}\}$ where θ is the azimuthal angle and $\underline{k}$ is the unit vector along the axis of symmetry. We assume that the meridian, C_o, generating the torus is a smooth curve of length 2L. A generic point P_o on C_o has the coordinates (r,θ,z). Also, the position of P_o is intrinsically determined by the arclength s measured along C_o. The configuration of the axial cross section of the undeformed torus in the θ-plane is shown in Fig. 3.9.1. It is plotted as the small closed curve C_o. The radial and axial distances of P_o are denoted by r and z respectively.

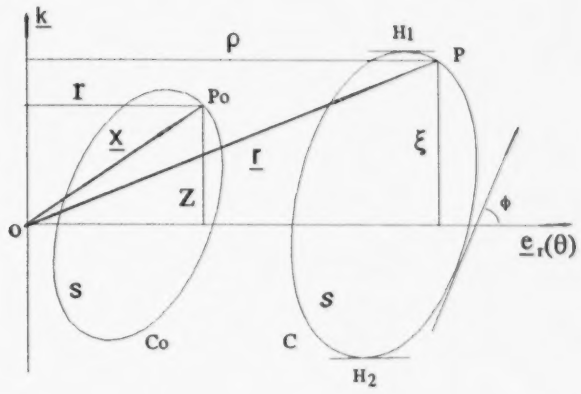

Fig. 3.9.1 Reference and Deformed torii

The position of point P_o on C_o is determined by

$$\underline{x} = r(s)\,\underline{e}_r(\theta) + z(s)\,\underline{k}, \qquad 0 \le \theta \le 2\pi,\ 0 \le s \le 2L \tag{3.9.1}$$

where $r(\cdot)$ and $z(\cdot)$ are regarded as functions of the arclength s.

We enforce the kinematic constraint that C_o does not intersect itself (no double points), thus such continuity conditions

$$r(0) = r(2L), \qquad z(0) = z(2L) \tag{3.9.2}$$

must be satisfied by functions r(s) and z(s) on C_o.

In an axisymmetric deformation, meridian C_o is carried by the deformation to the meridian C which generates the deformed torus. The point P_o is displaced to the point P on the deformed meridian C with the new coordinates (ρ,θ,ξ), where ρ and ξ are the radial and axial coordinates. The deformed axial section C referred to the same

orthogonal axes as the undeformed C_o was is shown in Fig 3.9.1. It is plotted as the large closed curve C. We define ϕ to be the angle between the tangent to C and the radial axis $\underline{e}_r(\theta)$. Here $0 \le \phi \le 2\pi$ for the closed C. H_1 and H_2 are the points of the crown where the tangents to C are horizontal.

The axisymmetric deformation of C_o is described by

$$\underline{r} = \rho(s)\, \underline{e}_r(\theta) + \xi(s)\, \underline{k}, \qquad 0 \le \theta \le 2\pi,\ 0 \le s \le 2L. \tag{3.9.3}$$

We assume that the deformed meridian C satisfies the same continuity as C_o. Thus the continuity conditions

$$\rho(0) = \rho(2L), \qquad \xi(0) = \xi(2L) \tag{3.9.4}$$

must be satisfied by functions $\rho(\cdot)$ and $\xi(\cdot)$ on C.

On close inspection, equations (3.9.1) and (3.9.3) are identical to those defined by equations (3.2.1) and (3.2.2) which describe the axisymmetric deformations of the surfaces of revolution. Therefore, kinematic relations derived in section 3.2 can be readily used in the formulation of torus problems. For the sake of notational consistency, the present notation is similar to that used in section 3.2. For example, λ, μ are used to denote the meridional and circumferential stretches.

Since the deformation is axisymmetric, the principal directions of the strain coincide with tangents to the deformed meridian and to lines of latitude, respectively. We denote them by the orthonormal vectors $\underline{l}$ and $\underline{m}$. With the aid of angle ϕ, the unit vector $\underline{l}$ can be expressed as

$$\underline{\ell} = \cos\phi\, \underline{e}_r(\theta) + \sin\phi\, \underline{k}, \qquad 0 \le \phi \le 2\pi \tag{3.9.5}$$

Use of (3.9.5) in $(3.2.11)_3$ yields two compatibility equations

$$\rho'(s) = \lambda \cos \phi, \quad \xi'(s) = \lambda \sin \phi. \tag{3.9.6}$$

which is different from relations in (3.2.13) by a minus sign in (3.2.13).

Using equation (3.9.6) in (3.2.21), we obtain

$$w_{\rho'} = w_\lambda \cos\phi, \quad w_{\xi'} = w_\lambda \sin\phi, \qquad 0 \le \phi \le 2\pi \tag{3.9.7}$$

We consider the internal pressure whose intensity p is prescribed. Since the deformed torus forms a closed surface, the potential energy of the pressure is given by (1.2.38). For the axisymmetric deformation defined by (3.9.3), the integrand of (1.2.38) is given in (3.2.15).

We assume that the membrane is homogeneous and isotropic possessing a strain energy function $w(\lambda,\mu)$ per unit area of reference surface. Hence, the total potential energy E[r] can be presented as a closed-path integral:

$$E[r] = 2\pi \oint_{C_o} H(\rho,\rho',\xi,\xi')ds \tag{3.9.8}$$

and $$H(\rho,\rho',\xi,\xi') = w(\rho,\rho',\xi') r - \rho (\rho \xi' - \xi \rho'). \tag{3.9.9}$$

Where ds is the element of arclength of the undeformed meridian C_o. The strain energy function $w(\lambda,\mu)$ is written as the form $w(\rho,\rho',\xi,\xi')$ through the substitution of λ and μ by their defining relations $(3.2.10)_{1,2}$. For convenience, we use $w(\cdot)$ to represent two different strain functions. They are distinguished by the explicit appearance of arguments.

The derivation of equilibrium equations is the same as it was in previous section and therefore will not be covered in detail. By setting the first variation of (3.9.8) to zero, the direct consequence results in the equilibrium equations

$$(r w_\lambda \cos\phi)' - w_\rho + p\rho\lambda\sin\phi = 0$$
$$(r w_\lambda \sin\phi)' - p\rho\rho' = 0 \qquad (3.9.10)$$

where relations (3.2.20) and (3.9.7) have been used. The prime denotes the differentiation with respect to the arclength s.

It is noted that the second of (3.9.10) can be integrated once to yield

$$r\, w_\lambda \sin\phi - \tfrac{1}{2}\, p\, \rho^2 = A \qquad (3.9.11)$$

where A is an integration constant to be determined.

It is worthwhile to examine some implications of equation (3.9.11) when it is evaluated at the points of the crown H_1 and H_2 where ϕ is equal to π and 0 respectively. By setting $\sin\phi = 0$, two real-valued roots of ρ can be solved from (3.9.11). Denoting them by ρ^h_1, ρ^h_2, respectively, we have

$$\rho^h_1 = (-2A/p)^{1/2}, \qquad \rho^h_2 = -(-2A/p)^{1/2} \qquad (3.9.12)$$

which are the radial coordinates of the upper and lower points of the crown on the deformed meridian *C*. This states that points of the crown are necessarily in vertical opposition on the deformed equilibrium configuration. This result must be valid for axisymmetric deformations of tori with arbitrary undeformed configurations, since the results in (3.9.12) are obtained without reference to the toroidal geometries. These results were also noted by Libai and Simmonds [3.32] in studying the axisymmetric deformations of toroidal surfaces. They claimed that conditions (3.9.12) can be realized only in the situation that the meridional stretch $\lambda \approx 1$ and the undeformed meridian C_o is near a circle. The conclusion was based on the restriction that the material particles at the crown points H_1 and H_2 on the deformed meridian *C* must be those near the

corresponding points of the crown on the undeformed meridian C_o. Their conclusion can be regarded as a particular example of (3.9.12). The constant A must be necessarily negative for $p > 0$ in order to have real values of ρ^h_1, ρ^h_2.

Carrying out the differentiation in equation $(3.9.10)_1$ and using the compatibility equation (3.9.6), we obtain a differential equation which is identical to the equations in (3.3.12). Equations (3.3.12), (3.3.7) and (3.9.12) constitute a system of first order differential-algebraic equations in λ, μ and ϕ.

The presence of the highly nonlinear strain energy function $w(\lambda, \mu)$ on the right-hand side of equations (3.3.12) makes it unfeasible to integrate them analytically, even if special forms of $w(\cdot)$ are chosen. We then have to resort to numerical solutions. To obtain the solution, we use the fact that $\dot{\lambda}$, $\dot{\mu}$ in (3.3.12), (3.3.7) are expressed explicitly as functions of λ, μ to transform the boundary-value problem to an initial-value problem. The problem becomes that of three-dimensional integration of the governing equations in the sense that three initial conditions must be guessed. For example, to integrate (3.3.12), (3.3.7) and (3.9.12) using an integration procedure such as the Runge-Kutta scheme, three variables λ, μ and ϕ must be known at a known point on C_o. Since $\phi(\cdot)$ is a geometrical quantity associated with the deformed configuration, it is not known *a priori* for an arbitrarily given reference meridian.

The numerical integration is anticipated to yield multiple solutions if all λ, μ and ϕ are to be guessed. It is necessary that a certain number of auxiliary conditions are supplemented to determine the solution.

Our interest is to consider some problems for which the initial conditions are obvious

and the number of auxiliary equations is the least. We are particularly interested in those reference geometries which possess a certain symmetry. For example, the reference meridian C_o is symmetric with respect to the horizontal axis $\underline{e}_r(\theta)$, such as a circular axial cross section considered by Kydoniefs [3.13]. If a torus with such symmetry is deformed by uniform internal pressure, the points of symmetry on C_o will be displaced to the corresponding points of symmetry on the deformed meridian C. Furthermore, ϕ at these points will be a known quantity before and after the deformation. Since any point on C_o can be chosen to be the origin for arclength, it can be set to zero at the points where ϕ is known. Hence the total number of unknown variables at the points of symmetry are reduced to variables, namely λ and μ. Further reduction in the number of unknowns seems difficult unless additional condition on the deformation is imposed. An example is that a point of symmetry is at the state of incipient wrinkling, i.e. $\mu = \upsilon(\lambda)$, in this case, only one initial guess λ is needed.

3.9.2 A tubular torus

The simplest reference toroidal membrane possessing the type of symmetry discussed in section 3.9.1 is a tubular torus. This is formed by two concentric circular tubes of the same height and are joined along their end boundaries. We denote by L, R^- and R^+ ($> R^-$) the height, inner and outer radii of the circular tubes. Thus, the axial cross-section of the torus is a rectangle of height L and width ($R^+ - R^-$). In particular, we consider the torus for which the width ($R^+ - R^-$) = δ is approximately zero. The undeformed configuration may be regarded as a collapsed circular tube with radius R (=

$R^- = R^+$).

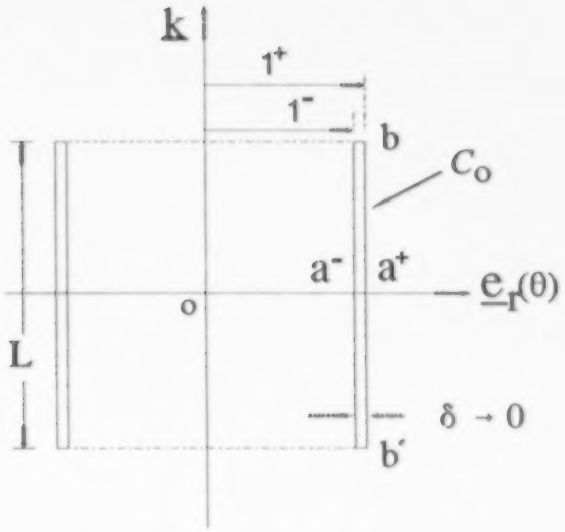

Fig. 3.9.2 A tubular torus

If the material constant G and the reference radius R are chosen to be the non-dimensionalization parameters in (3.3.5), the nondimensional radii are $r = 1^-, 1^+$. The axial cross-section of the nondimensional reference torus is shown in Fig. 3.9.2. The middle points of the inner and outer tubes are denoted by a^- and a^+, while the upper and bottom ends are denoted by b and b´.

In addition to (3.3.5), the nondimensional constant A in (3.9.12) is

$$\hat{A} = A / GR. \tag{3.9.13}$$

All further reference to this quantity is with respect to the nondimensional form but the hat is dropped.

Suppose that the torus is unstrained in the reference state. We speculate that for a

certain amount of internal pressure the torus will change its shape markedly before coming to equilibrium. Accompanying the deformation, the inner middle point a^- is anticipated to move inwardly toward axis of symmetry along $\underline{e}_r(\theta)$ by a finite distance. As a result, in the neighbourhood of a^-, $\lambda^- > 1$, but μ^- is considerably smaller than unity and thus smaller than the natural width $\upsilon(\lambda^-) = 1/(\lambda^-)^{1/2}$. λ^-, μ^- denote the meridional and circumferential stretches at a^-. According to criterion (1.3.17), the membrane is circumferentially wrinkled in a certain region containing a^-. By contrast, the internal pressure is expected to force the outer middle point a^+ to move outwardly along $\underline{e}_r(\theta)$. In a region near and including a^+, $\lambda^+ > 1$, $\mu^+ > 1$, where λ^+, μ^+ denote the meridional and circumferential stretches at a^+. Under this circumstance, the membrane is tense in this region. Moreover, by symmetry of the deformation, the vertical tangents at a^-, a^+ on C_o will remain subsequently vertical on the deformed meridian C.

We also speculate that the inward movement of a^- continues as the pressure increases until it approaches a certain value beyond which a^- starts to move outwardly. As the pressure increases, the size of wrinkled region reduces and, eventually, the membrane becomes entirely tense. However, the numerical results we have obtained only reflect the outward movement of a^- as the pressure increases. We infer that the pressure range we have considered is too large to contain that region which corresponds to the inward movement.

Since the reference configuration and the subsequent deformations are symmetric with respect to the plane perpendicular to the axial axis $\underline{k}$, it is sufficient to consider the

deformation of the lower half of the torus. The axial cross section of the deformed torus is sketched in Fig. 3.9.3.

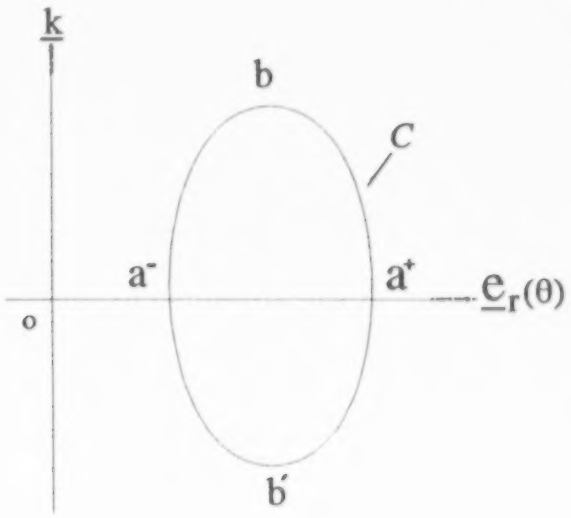

Fig.3.9.3 Diagram of an inflated tubular torus

3.9.3 Governing equations, boundary conditions and solving variables

Although the governing equations are formulated to include tori with arbitrary configurations, the emphasis in this section will be on results corresponding to a tubular torus discussed in section 3.9.2.

As shown in Fig.3.9.2, the deformations of a tubular membrane can be regarded as the deformations of two circular tubes. The general solutions are constructed by letting the solution of each tube satisfy continuity conditions at the joint points b′ and b. The equilibrium equations corresponding to the inflation of the tubes are identical to those derived in section 3.3. For the circular tubes as shown in Fig.3.9.2, the reference radius

$r(s) = 1$ is a constant, it follows that $r'(s) = 0$. Using this result in (3.3.12) and (3.3.7), we reduce the equations to

$$\begin{cases} \dot{\lambda} = \dfrac{(w_{\mu} - \lambda w_{\lambda\mu})\cos\phi}{w_{\lambda\lambda}} \\ \dot{\mu} = \lambda\cos\phi \\ \dot{\xi} = \lambda\sin\phi \\ \sin\phi = \dfrac{(\frac{1}{2}p\mu^2 + A)}{w_{\lambda}} \end{cases} \tag{3.9.14}$$

in which the relation $\mu = \rho$ has been used. We also include equations $(3.3.6)_2$ and (3.9.12) here as $(3.9.15)_{3,4}$ to complete the system of solving equations.

Since we only consider the deformation of the lower half of the torus, we set the origin of the arclength, i.e. $s = 0$ at a^-. According to Fig. 3.9.3, the boundary conditions can be summarized as follows

$$\phi = 3\pi/2,\ \xi(0) = 0,\ \lambda = \lambda^-,\ \mu = \mu^-,\ s = 0, \qquad \text{at } a^- \tag{3.9.15}$$

$$\phi = \pi/2,\ \xi(L) = 0,\ s = L, \qquad \text{at } a^+ \tag{3.9.16}$$

where λ^- and μ^- need to be prescribed. L is the height of the tube as shown in Fig. 3.9.2.

As we noted early, the initial-value problem involves more than one initial parameters. In what follows, we demonstrate that the solution can be formulated in terms of λ^- and μ^-. It is interesting to note that the height of reference tube may be calculated based on λ^- and μ^-, instead of being given a priori.

Since $\phi = 3\pi/2$ at a^-, thus $\sin\phi = -1$, for the given initial values λ^-, μ^-, A is determined by the (3.9.14)

$$A = - w_\lambda(\lambda^-,\mu^-) - \tfrac{1}{2}\, p\, (\mu^-)^2 \tag{3.9.17}$$

where $A < 0$ is ensured by conditions $w_\lambda(\lambda^-,\mu^-) \geq 0$, $p \geq 0$.

At a^+, $\phi = \pi/2$, $\sin\phi = +1$. Values λ^+, μ^+ of λ and μ at a^+ must satisfy $(3.9.14)_4$. We have

$$A = w(\lambda^+, \mu^+) - \tfrac{1}{2}\, p\, (\mu^+)^2 \tag{3.9.18}$$

For the axisymmetric deformations of the surface of revolution, Pipkin [3.15] has derived an equation corresponding to the first integral of equilibrium equations, this is presented in the form

$$w(\lambda,\mu) - \lambda\, w_\lambda(\lambda,\mu) = c \tag{3.9.19}$$

where c is a constant along the reference meridian C_o. This equation can be used to derive $(3.9.14)_1$ with the help of $(3.9.14)_2$. Equation (3.9.19) is valid at each point on C_o, it must be satisfied by λ and μ at a^-, a^+, respectively. Hence, we have

$$w(\lambda^-,\mu^-) - \lambda^-\, w_\lambda(\lambda^-,\mu^-) = c \tag{3.9.20}$$

$$w(\lambda^+,\mu^+) - \lambda^+\, w_\lambda(\lambda^+,\mu^+) = c \tag{3.9.21}$$

A solution satisfying conditions (3.9.17)-(3.9.21) determines a deformed meridian which has vertical tangents at a^- and a^+, respectively. So far, however, the solution is obtained without being required to satisfy the boundary condition $(3.9.16)_2$. As a result, it is possible that, for some initial guesses λ^- and μ^-, the vertical tangency points a^- and a^+ might not be on the same horizontal axis $\underline{e}_r(\theta)$. To impose the restriction of symmetry that a^- and a^+ are on the axis $\underline{e}_r(\theta)$ on the deformed torus, i.e. $\xi^- = \xi^+ = 0$, where $\xi^- = \xi^+$ are the axial desplacements of the points a^- and a^+, following constraint must be satisfied

$$\int_0^L d\xi = 0. \tag{3.9.22}$$

which is the integration of $(3.9.14)_3$ together with boundary conditions $(3.9.15)_2$ and $(3.9.16)_2$. In view of $(3.9.14)_{2,3}$, (3.9.22) can be written as a function

$$g(\lambda^-, \mu^-) = \int_{\mu^-}^{\mu^+} \tan\phi \, d\mu = 0 \tag{3.9.23}$$

where ϕ is the function of μ through $(3.9.14)_4$ in which λ is expressed in terms of μ as the solution to (3.9.19). Equation (3.9.23) restricts that values of λ^- and μ^- corresponding to an equilibrium symmetric deformation can not be arbitrary but satisfy the constraint relation.

If it is possible to eliminate λ^- from (3.9.23) in terms of μ^-, the problem is thus reduced to the one parameter problem. In principle, the remaining quantities λ^+, μ^+, A and c can also be expressed in terms of μ^- as solutions to equations (3.9.17,18) and (3.9.20,21), i.e.

$$\lambda^+ = f_1(\mu^-; p) \tag{3.9.24}$$

$$\mu^+ = f_2(\mu^-; p) \tag{3.9.25}$$

$$A = f_3(\mu^-; p) \tag{3.9.26}$$

$$c = f_4(\mu^-; p) \tag{3.9.27}$$

$$\lambda^- = f_5(\mu^-; p) \tag{3.9.28}$$

provided that the simultaneous equations are solvable analytically. The dependence on the pressure p is obvious. $f_i(\cdot)$ are some nonlinear functions of μ^- and p.

The hight of the tube can be obtained by integrating $(3.9.14)_2$. It is

$$L = \int_{\mu^-}^{\mu^+} \frac{du}{f(\mu;c)\sqrt{1 - \frac{[\frac{1}{2}p\mu^2 + A]^2}{w_\lambda[f(\mu;c)]^2}}} \qquad (3.9.29)$$

where $f(\mu;c)$ is the solution of λ from equation (3.9.19). In view of (3.9.24)-(3.9.28), L is a certain function of μ^- and p, i.e. $L = f_6(\mu^-; p)$. Upon close inspection, equations $(3.9.14)_{1,2}$ are characterized by the fact that $\dot{\lambda}$, $\dot{\mu}$ do not depend on the arclength s explicitly. This special feature can be utilized to calculate L as a sum of total stepsizes from a^- to a^+. We used this scheme in our numerical integrations.

Given the nature of high nonlinearity associated with equations (3.9.17, 18) and (3.9.20, 21), multiple μ^- are to be expected as the solutions of the simultaneous equations. It is reasonable to speculate that all these μ^- satisfying the constraint equation (3.9.28) will yield an identical height L based on (3.9.26). In other words, the problem may be posed as that of possessing multiple solutions of deformation for a prescribed pressure applied on a given tubular torus. To determine the solution, we calculate the potential energy associated with each solution and conjecture that the solution with the smallest potential energy is physically realistic. Same procedure was used in section 3.4 to resolve the non-uniqueness of solution of the inflation of a hemispherical membrane fixed at the equator.

3.9.4 The solution procedure

In practice, however, solutions (3.9.24)-(3.9.28) which give explicit dependence of λ^+,

μ^+, A, c and λ^- on μ^- may not be obtainable analytically due to the complex form of nonlinear equations (3.9.17, 18), (3.9.20, 21) and (3.9.23). These equations may be solved numerically using iterative method such as Newton-Raphson iterations, but it involves more initial guesses for λ^+, μ^+, A, c and λ^- and solving these equations itself becomes more complicated than problem originally posed.

Under this circumstance, it is practically feasible to integrate the governing equations numerically as an initial-value problem. The governing equations expressed in (3.9.14) are nonlinear partial differential equations of first order and they require no special effort for the numerical integration method to be implemented. In principle, this should constitute a straightforward numerical integration exercise which may be carried out using a fourth-order Runge-Kutta forward marching method as was used in previous sections. For an initial-value problem procedure to become readily applicable, a sufficient number of initial conditions must be specifies. In this case, there are two initial conditions to consider. For example, both μ^- and λ^- must be specified at a^- in order to start the integration at a^-. Moreover, some kind of an iterative scheme must simultaneously be implemented to adjust the initial guesses in order for the boundary conditions (3.9.15, 16) to be satisfied by the solution.

We construct the initial guesses for μ^- and λ^- in the following manner: Let $\mathscr{F}$: $\{\mu^-_i \,|\mu^-_j > 0, i=1,2,3,..,M\}$ and $\mathscr{E}$: $\{\lambda^-_j \,|\lambda^-_j > 1, j=1,2,3,..,N\}$ be the sets containing the real values of μ^- and λ^-, respectively. The initial guess, denoted by $\mathcal{P}_{ij}$, is formed by pairing the i-th member of $\mathscr{F}$ which is μ^-_i with the j-th member of $\mathscr{E}$ which is λ^-_j, so that $\mathcal{P}_{ij} = \{\mu^-_i, \lambda^-_j\}$ is the ij-th member of $\hbar$ which constitutes the set containing all

possible initial guess pairs based on $\mathcal{F}$ and $\mathcal{E}$, i.e. $\hbar = \mathcal{F} \times \mathcal{E}$: $\{\mathcal{P}_{ij}, i=1,2,3,..,M, j=1,2,3,..,N\}$. The initial guess set $\hbar$ should contain a total of M × N pairs of $\mathcal{P}_{ij}$. Note that each member of $\mathcal{F}$ must be greater than zero (>0), while each member of $\mathcal{E}$ must be greater than unity (>1). The domain of $\hbar$ is shown as the shaded area in Fig.3.9.4

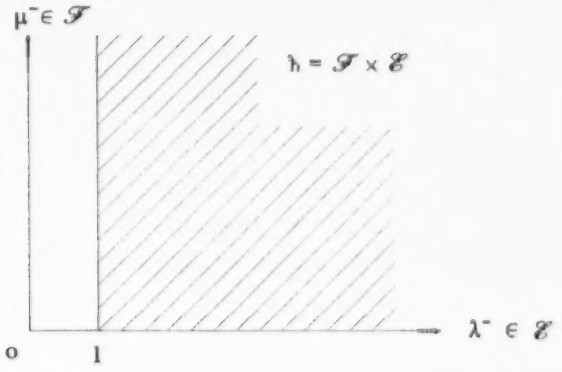

Fig. 3.9.4 Domain of $\hbar$ in λ^{-}-μ^{-} plane.

For the initial guess $\mathcal{P}_{ij} \in \hbar$, the constant A is calculated from equation (3.9.17). Note that $\phi = 3\pi/2$ at a^{-}. In addition, to check the accuracy of the solution based on $\mathcal{P}_{ij}$, the constant c in the first integral (3.9.19) is computed. Boundary conditions $(3.9.15)_{2,5}$ are used as the initial conditions at a^{-}, while boundary conditions $(3.9.16)_{1,2}$ are used as two shooting conditions at a^{+}. Using a Runge-Kutta forward marching integration method, the governing equations (3.9.14) are integrated up to a point, namely the end-point, where the first shooting condition $(3.9.16)_1$ is satisfied. Satisfaction of condition $(3.9.16)_1$ implies that the tangent to the deformed meridian at the end-point is vertical, i.e. $\phi = \pi/2$. This end-point should correspond to the outer middle point a^{+} on the

reference configuration. If the initial guess $\mathcal{P}_{ij}$ contains the actual values of μ^- and λ^- corresponding to an equilibrium state, ξ^+, the value of ξ yielded by integration at the vertical tangency point a^+ must be zero to satisfy the second shooting condition $(3.9.16)_2$. If $\xi^+ = 0$, the height L of the reference torus is calculated from (3.9.29), or alternatively, it can be obtained as the sum of the total shooting step-size Δs, i.e. $L = \sum \Delta s$. The total potential energy can also be calculated from the integral (3.9.8) by the commonly used Simpson's integration scheme.

For the initial guess $\mathcal{P}_{ij}$ which is different from the actual value, the point a^+ ($\phi = \pi/2$) thus calculated may have the value ξ^+ which is different from zero. For example, in the case $\xi^+ > 0$, the end-point a^+ having the vertical tangent is above the horizontal axis $\underline{e}_r(\theta)$. In this case, an iterative shooting scheme must be employed to find the solution. To this end, the initial guess is changed from $\mathcal{P}_{ij}$ to $\mathcal{P}_{ij+1}$ by a small increment and the integration procedure is repeated until the condition $(3.9.16)_2$ is satisfied by the vertical tangency point a^+. At this stage, only λ^-_j is changed to λ^-_{j+1} by a small increment while μ^-_i remains the same. If such solutions can not be found for all members of $\lambda^-_j \in \mathcal{E}$, we change μ^-_i to μ^-_{i+1} to form a new initial guess $\mathcal{P}_{i+1j} = \{\mu^-_{i+1}, \lambda^-_j, j=1,2,3,\ldots,N\}$ and repeat the forgoing solution procedure until the solution is found. For completeness, all members of the initial guess set $\hbar$ should be considered. The height L and the total potential energy $E[\cdot]$ associated with each solution should also be calculated.

Not every member $\mathcal{P}_{ij} \in \hbar$ leads to a solution of the governing equations. Those $\mathcal{P}_{ij}$ which are solutions constitute a subset of $\hbar$, namely $\mathcal{P}^s = \{\mathcal{P}_{ij}, i=1,2,3,\ldots,M_s, j=1,2,3,\ldots,N_s\}$ where $M_s < M$ and $N_s < N$. There also exists a set $\mathcal{L} = \{L_{ij},$

$i=1,2,3,\ldots,M_s$, $j=1,2,3,\ldots,N_s\}$ which contains the height of the reference tube corresponding to the solution $\mathcal{P}_{ij} \in \mathcal{P}^s$. It is verified later and displayed in Fig.3.9.17 that, for a fixed μ^-_i, there exists a one-to-one correspondence between L_{ij} (fixed i) and λ^-_j for $j = 1$ to N_s. Furthermore, the results also show that $L_{ij} \neq L_{ij+1}$ (fixed i) when $\lambda^-_j \neq \lambda^-_{j+1}$. Similarly, for a fixed λ^-_j, a uniqueness relation is found between L_{ij} (fixed j) and μ^-_j for $i = 1$ to M_s.

If the values of the members of $\mathcal{L}$ are distinct, a particular reference height L and the initial guess $\{\mu^-,\lambda^-\}$ corresponding to a solution can be read off from the sets $\mathcal{L}$ and $\mathcal{P}^s$ simultaneously. However, situation becomes complicated when values of some members $L_{ij} \in \mathcal{L}$ are equal. For example, let $\mathcal{P}_{km}$ and $\mathcal{P}_{ln}$ be two members of $\mathcal{P}^s$, the solution may yield the result that L_{km} associated with $\mathcal{P}_{km}$ and L_{ln} associated with $\mathcal{P}_{ln}$ are equal, i.e. $L_{km} = L_{ln}$. The fact that $\mathcal{P}_{km} \neq \mathcal{P}_{ln}$ leads to $\mu^-_k \neq \mu^-_l$ and $\lambda^-_m \neq \lambda^-_n$, therefore, the problem can be stated as that for the reference torus L_{km} $(=L_{ln})$, there are two distinct equilibrium configurations corresponding to $\mathcal{P}_{km}$ and $\mathcal{P}_{ln}$, respectively. As shown later, three possible solutions exist for a given pressure and the reference torus. We resolve the non-uniqueness issue using the minimum energy principle.

Let $\mathcal{L}^s$ be the subset of $\mathcal{L}$, namely $\mathcal{L}^s = \{L_k,\ k=1,2,3,\ldots,S\}$ containing certain members of $\mathcal{L}$ and each element of $\mathcal{L}^s$ corresponds to multiple initial guesses $\mathcal{P}_{ij} \in \mathcal{P}^s$. We suppose that L_k has m initial guesses $\{\mu^-,\lambda^-\}_{t=1,2,3,m}$ and use our early solution-conjecture to determine the solution. We, then, compute the total potential energy associated with each solution $\{\mu^-,\lambda^-\}_t$. The physically realistic solution $\{\mu^-,\lambda^-\}$ should possess the smallest $E[\cdot]$

$$E[\mu^-,\lambda^-] < E[\{\mu^-,\lambda^-\}_t], \quad t = 1,2,3,\ldots,m. \tag{3.9.30}$$

where $E[\cdot]$ is calculated from (3.9.8). The solution $\{\mu^-,\lambda^-\}$ can be determined by straightforward energy comparison.

The forms of the strain energy function $w(\cdot)$ and the relaxed strain energy function $w^R(\cdot)$ must be specified in order to obtain explicit solutions. We assume that the membrane is composed of Ogden's material defined by (3.1.3). The relaxed strain energy $w^R(\cdot)$ is given in (3.1.9) for the natural width defined by relation (3.1.8).

The intensity of the internal pressure and the height of the reference torus certainly have an influence on the state of deformation. Two states of deformation are examined in this context, namely fully tense and partly wrinkled deformations. Calculations have been considered for these two states separately.

3.9.5 Fully tense solution

We first discuss the solution procedure for the membrane which is fully tense after the deformation. A suitable non-dimensional step-size shall be chosen in order to use the Runge-Kutta algorithm. It is noteworthy that the differential equations (3.9.14) are independent of arclength, the choice of step-size will not change the solution itself. In other words, the length L calculated by accumulating the step-size will not be affected by the actual size of the step-size, i.e. $L = \sum \Delta s^1 = \sum \Delta s^2, \ldots, = \sum \Delta s^k$ where the sup-1,2,3,...k denotes the different choice of step-size. However, the choice of Δs does affect the accuracy of solutions.

Equations (3.9.14) are valid at each point along the reference meridian C_o. According

to the boundary condition $(3.9.15)_1$, $\phi = 3\pi/2$ at a^-, using this condition in equations $(3.4.19)_{1,2}$ shows that both $\dot{\lambda}$ and $\dot{\mu}$ are zero. As a consequence, the Runge-kutta algorithm ceases to start at a^-. The integration algorithm must be started at a point, namely a_1, which is slightly away from a^- by a small increment Δs. Values λ_1, μ_1 of λ, μ at a_1 can be calculated using Taylor's expansion based on the non-zero derivatives of λ, μ of higher orders at a^-. In this case, we only retain the second derivatives in Taylor's expansion. The results are valid approximates provided $|\Delta s| << 1$. After differentiating equations $(3.9.14)_{1,2}$ and evaluating the results at a^-, we obtain

$$\ddot{\lambda}_{a^-} = \frac{(w_\mu - \lambda w_{\lambda\mu})(p\mu\lambda + w_\lambda)}{w_{\lambda\lambda} w_\lambda}\Big|_{a^-} \tag{3.9.31}$$

and

$$\ddot{\mu}_{a^-} = \frac{\lambda(p\mu\lambda + w_\mu)}{w_\lambda}\Big|_{a^-} \tag{3.9.32}$$

Thus, the Taylor's expansion determines

$$\begin{aligned} \lambda_1 &= \lambda^- + \tfrac{1}{2}\ddot{\lambda}(\Delta s)^2 \\ \mu_1 &= \mu^- + \tfrac{1}{2}\ddot{\mu}(\Delta s)^2 \end{aligned} \tag{3.9.33}$$

To seek the fully tense deformation, the initial guess $\mathcal{P}_{ij} = \{\mu^-_i, \lambda^-_j\}$ must be given such that condition $\mu^-_i > \upsilon(\lambda^-_j)$ must be satisfied. Here $\upsilon(\lambda^-_j) = 1/(\lambda^-_j)^{1/2}$ is the natural width evaluated at a^-. With a small step-size Δs, values of λ_1 and μ_1 for the given $\mathcal{P}_{ij}$ are obtained from (3.9.31) with the aid of (3.9.32,33). The constant A based on $\mathcal{P}_{ij}$ is determined from (3.9.17). In turn, λ_1, μ_1 and A are used in $(3.9.14)_4$ to start the integration at a_1. As the integration proceeds, the reference height L is calculated as the

summation of the total integration step-size. To check the suitability of the given $\mathcal{P}_{ij}$, the value of $\sin\phi$ is calculated at each mesh point. The function $\sin\phi$ must be a monotonically increasing function of the arclength s. The integration procedure is terminated when $\sin\phi = 1$. The remaining solution procedure which includes modifying the initial guess and energy comparison is described in section 3.9.4.

3.9.6 Partly wrinkled solution

For a certain range of the pressure, wrinkling will start at a^- and spreads along the deformed meridian. Thus the deformed torus is partly wrinkled in an inner region containing the point a^-. Equilibrium equations in the wrinkled region are obtained by using the relaxed strain energy function $w^R(\lambda)$ in the place of $w(\cdot)$ in equations (3.9.14). Since $w^R(\lambda)$ depends only on the meridional stretch λ, (3.9.14) are further reduced to the form

$$\begin{cases} \dot{\lambda} = 0 \\ \dot{\mu} = \lambda\cos\phi \\ \dot{\xi} = \lambda\sin\phi \\ \sin\phi = \dfrac{(\frac{1}{2}p\mu^2 + A)}{w_\lambda^R} \end{cases} \tag{3.9.34}$$

Since $\phi = 3\pi/2$ at a^-, $\sin\phi = -1$. For the given initial values μ^- and λ^-, A is determined by $(3.9.34)_4$ such that

$$A = - w^R{}_\lambda(\lambda^-) - \tfrac{1}{2}\, p\, (\mu^-)^2 \tag{3.9.35}$$

where $A < 0$ is ensured by conditions $w^R{}_\lambda(\lambda^-) \geq 0$, $p \geq 0$.

It is clear from $(3.9.34)_1$ that λ is a constant along the meridian in the wrinkled region, i.e. $\lambda = \lambda^-$. The arclength corresponding to the wrinkled range can be calculated in terms of λ^-. We use s_w to denote the arclength corresponding to the wrinkled region, and μ_w to denote the value of μ at s_w. Since λ is a constant in the wrinkled region, we have $\mu_w = v(\lambda^-) = 1/(\lambda^-)^{1/2}$ at s_w. From $(3.9.34)_{2,4}$, we have

$$s_w = \int_{\mu^-}^{\mu_w} \frac{du}{\lambda^- \sqrt{1 - \dfrac{[\frac{1}{2}p\mu^2 + A]^2}{[w_\lambda^R(\lambda^-)]^2}}} \tag{3.9.36}$$

Since μ_w $(= 1/(\lambda^-)^{1/2})$ and λ_w $(= \lambda^-)$ are known at s_w, we can use these values as the initial guesses to integrate the governing equations (3.9.14) in the tense region $(L - s_w)$. The solution follows the same procedure as described in section 3.9.5.

To search for solutions of the membrane that is partly wrinkled, the initial guess $\mathcal{P}_{ij} = \{\mu^-_i, \lambda^-_j\}$ must be specified such that $\mu^- < 1/(\lambda^-)^{1/2}$. For the given $\mathcal{P}_{ij}$, the constant A is determined from equation (3.9.35). The arclength for the wrinkled region s_w is calculated from (3.9.36). The total height of reference torus is obtained as the addition of s_w and the accumulation of the total shooting step-size in the tense region.

3.9.7 Numerical results

Numerical examples are presented for non-dimensional pressures p = 2.4, 2.8, 3.2, 3.6, 4.0, 4.4. For the range of initial guess $\{\mu^-, \lambda^-\}$ that we have considered, we read off solutions for L = 0.3, L = 0.5 and L = 1.0 from $\mathcal{L}^s$.

Non-uniqueness of solution has been found. To show this, we present the numerical

solutions for L = 0.5 in Table 3.9.1. In particular, for p = 2.4, the results are given graphically in Fig. 3.9.15, 16, 17. There are three solutions associated with each pressure p. For example, for p = 2.4, $\{\mu^-,\lambda^-\}$ = {0.811, 1.1486}, {1.71, 3.58} and {3.35, 6.40} respectively. The total potential energy associated with each solution are E_1 = -0.04577, E_2 = 0.8195 and E_3 = 0.22959. By energy comparison, solution $\{\mu^-,\lambda^-\}$ = {0.811, 1.1486} possesses the minimum total potential energy. This is the solution which furnishes the smallest $\{\mu^-,\lambda^-\}$ with respect to other two high-energy solutions. For this solution, the membrane is partly wrinkled in an area near a^-. Whereas, the deformations possessing higher energies are fully tense. The deformed axial sections associated with each solution are plotted in Fig.3.9.5. Dashed line indicates the wrinkled region.

Fig. 3.9.6 shows the deformed axial sections for p = 2.4, 2.8, 3.2, 3.6, 4.0, 4.4. It may be seen that as the pressure increases, the point a^- moves outwardly along $\underline{e}_r(\theta)$. Meanwhile, the wrinkled region decreases by converging toward a^-. The wrinkled region disappears and the entire membrane is fully tense at p = 4.0, 4.4. Fig.3.9.7 shows the distribution of λ along the undeformed meridian C_o. Straight lines represent the constant λ in the wrinkled region. It is clear that λ increases with increasing pressure. Fig. 3.9.7 also shows that at low pressures, λ decreases monotonically from the inner point a^- to the outer point a_+. As the pressure increases, this monotonicity seems to disappear as λ near the outer point a^+ increases. We again observe the discontinuities in λ at the boundary s_w. To verify our wrinkling hypothesis, in Fig. 3.9.8, we plot the distributions of the circumferential stretch μ and the associated natural

width $v(\lambda) = 1/\lambda^{1/2}$ along C_o. For simplicity, we plot the ratio $\mu/v(\lambda)$. It is verified that in wrinkled regions, $\mu/v(\lambda) < 1$, or $\mu < v(\lambda)$, and $\mu/v(\lambda) > 1$ in tense regions.

The reference configuration affects the state of deformation. For a given range of pressures, we speculate that tall tubes are likely to be wrinkled than short tubes. To verify this assumption, we plot the profiles of the deformed axial cross sections of torii with height L = 0.3 and L = 1.0 in Fig 3.9.9 and Fig. 3.9.12 respectively. It is shown that the torus with L = 1.0 are partly wrinkled for all the pressures we have considered, whereas, for the short torus with L = 0.3, it remains fully tense for the same pressures. By comparing Fig.3.9.9 and Fig.3.9.12, the tall torus has large wrinkled region than that in the short torus in the same pressure range. The distributions of λ and ratio $\mu/v(\lambda)$ for L = 0.3 are displayed in Fig. 3.9.10 and Fig. 3.9.11 respectively. Similar quantities are plotted in Fig. 3.9.13, Fig. 3.9.14. for L = 1.0.

We have stated previously that not every initial guess we have prescribed leads to a solutions of the governing equations. The range for initial guess is not known a priori and it is determined on the trial-and-error basis by numerical experiments. We demonstrate this by considering the solutions for p = 2.4. As shown in Fig. 3.9.15, the numerical results show that the initial guess μ^- must be in the range (0.712, 4.80) in order to obtain solutions with $\lambda^- > 1.0$. This is obtained by testing every initial guess pair $\{\mu^-,\lambda^-\}$ for $\mu^- \in (0, 10)$ and $\lambda^- \in (1.0, 12)$. Particularly, when $\mu^- \in (0.712, 1.0)$, the solution range of λ^- is (1.0, 2.0). When $\mu^- \in (1.0, 2.0)$, solutions can only be found for $\lambda^- \in (2.0, 4.0)$. As μ^- is increased to $\mu^- \in (2.0, 4.0)$, solutions are found for $\lambda^- \in (4.0, 8.0)$. In the last range we have considered, $\mu^- \in (4.0, 8.0)$,

solutions can be found for both $\lambda^- \in (4.0, 8.0)$ and $\lambda^- \in (8.0, 12.0)$.

Fig. 3.9.17 shows that solutions of μ^- and λ^- are not continuous, solution curve $\{\mu^-, \lambda^-\}$ is formed by the set of discrete points. For example, the first solution is found at $\mu^- = 0.712$, $\lambda^- = 1.3169$, whereas the second solution is found at $\mu^- = 0.721$, $\lambda^- = 1.2872$. Both μ^- and λ^- jumps by a finite value. It is also worth noting that, when $\mu^- \in (0.712, 1.0)$, two solutions of λ^- are found for one μ^-. Similar relation is also encountered when $\mu^- \in (4.68, 4.80)$. In the range that $\mu^- \in (1.0, 4.68)$, only one solution λ^- is found for each μ^-. Fig. 3.9.15 and Fig. 3.9.16 show that μ^--L and λ^--L curves are also formed by the sets of discrete solution points.

As illustrated in Fig.3.9.15, there is a certain non-uniqueness in the relation between the initial guess μ^- and the height of reference tube L at fixed internal pressure p. More particularly, in the range $0.452 < L < 1.0$, a given L is associated with three different values of μ^-. For $L < 0.452$ and $L > 1.0$, the relation between L and μ^- is unique. By contrast, Fig.3.8.17 shows that the relation between L and λ^- is one-to-one for fixed μ^-.

The solutions for $L = 0.5$ and $p = 2.4$ are obtained by reading off values of μ^- and λ^- from Fig.3.9.16 and Fig.3.9.17 simultaneously.

p=2.4	μ^-	0.811	1.71	3.35
	λ^-	1.149	3.58	6.40
p=2.8	μ^-	0.82	1.40	4.28
	λ^-	1.178	2.86	7.68
p=3.2	μ^-	0.83	1.23	4.96
	λ^-	1.228	2.42	8.60
p=3.6	μ^-	0.84	1.12	5.44
	λ^-	1.287	2.14	9.36
p=4.0	μ^-	0.849	1.03	6.0
	λ^-	1.991	1.32	10.08

Table 3.9.1 Multiple solutions of μ^- and λ^- for L = 0.5.

p	μ^-	λ^-	E
2.4	0.8125	1.1455	-0.04577
2.8	0.8225	1.1820	-0.03879
3.2	0.8314	1.2265	-0.03326
3.6	0.8355	1.2800	-0.03227
4.0	0.8473	1.3540	-0.02770
4.4	0.8652	1.4340	-0.02326

Table 3.9.2 Solutions of μ^- and λ^- and the potential energy for L = 0.5.

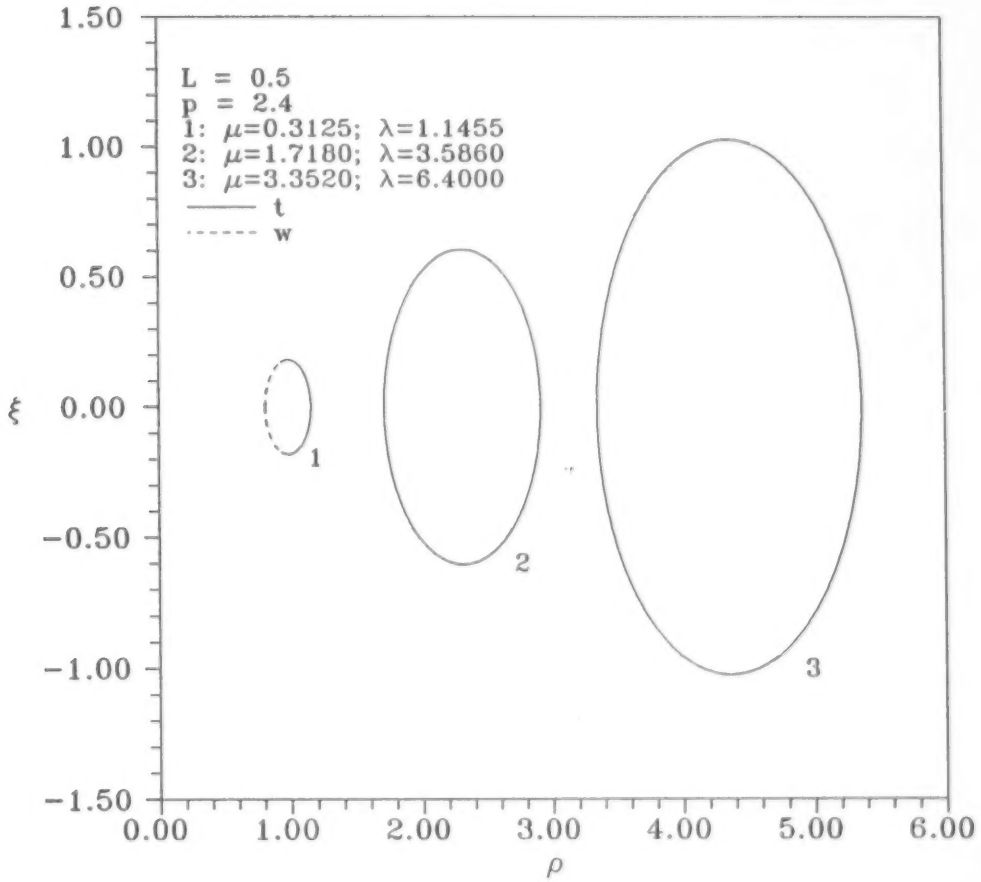

Fig.3.9.5 Deformed axial cross-sections

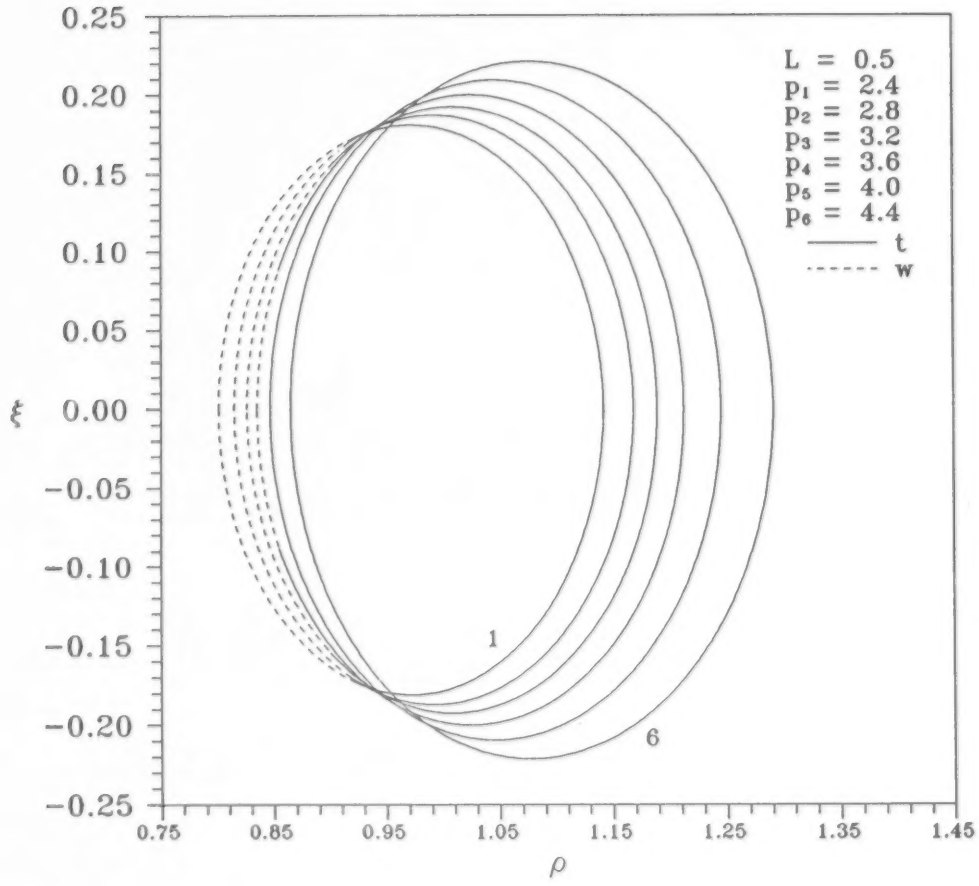

Fig. 3.9.6 Deformed axial cross-sections

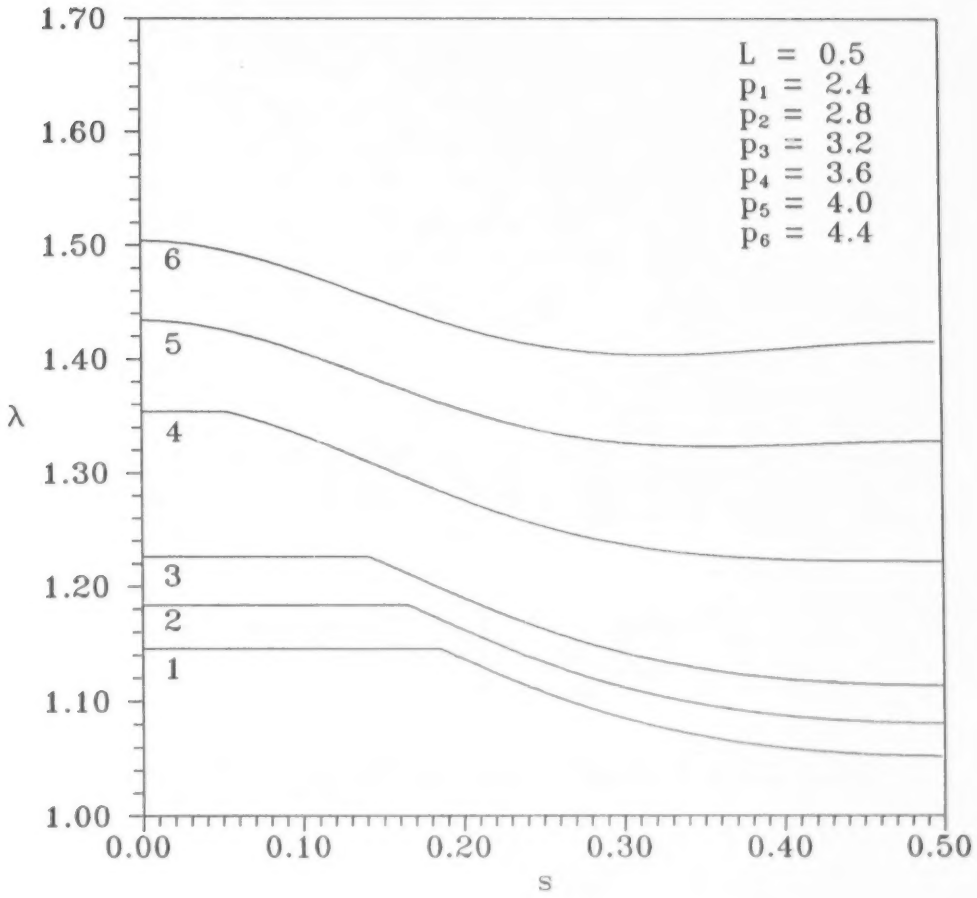

Fig. 3.9.7 Distribution of λ

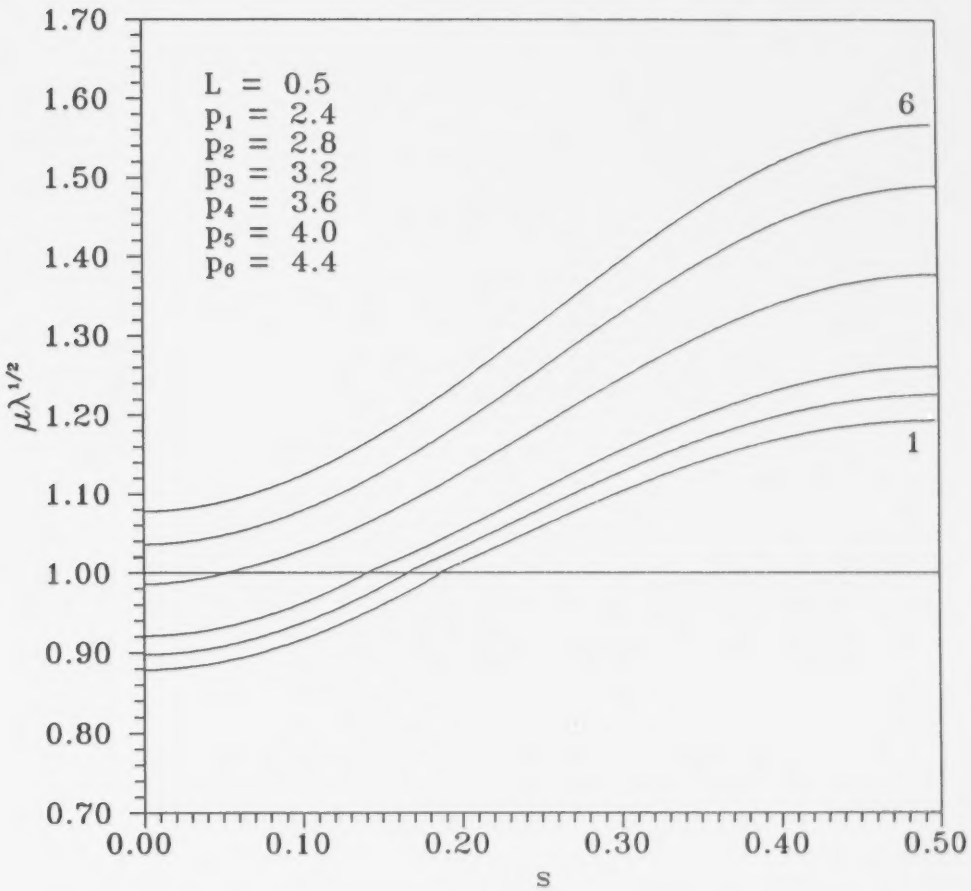

Fig. 3.9.8 Distribution of $\mu\lambda^{1/2}$

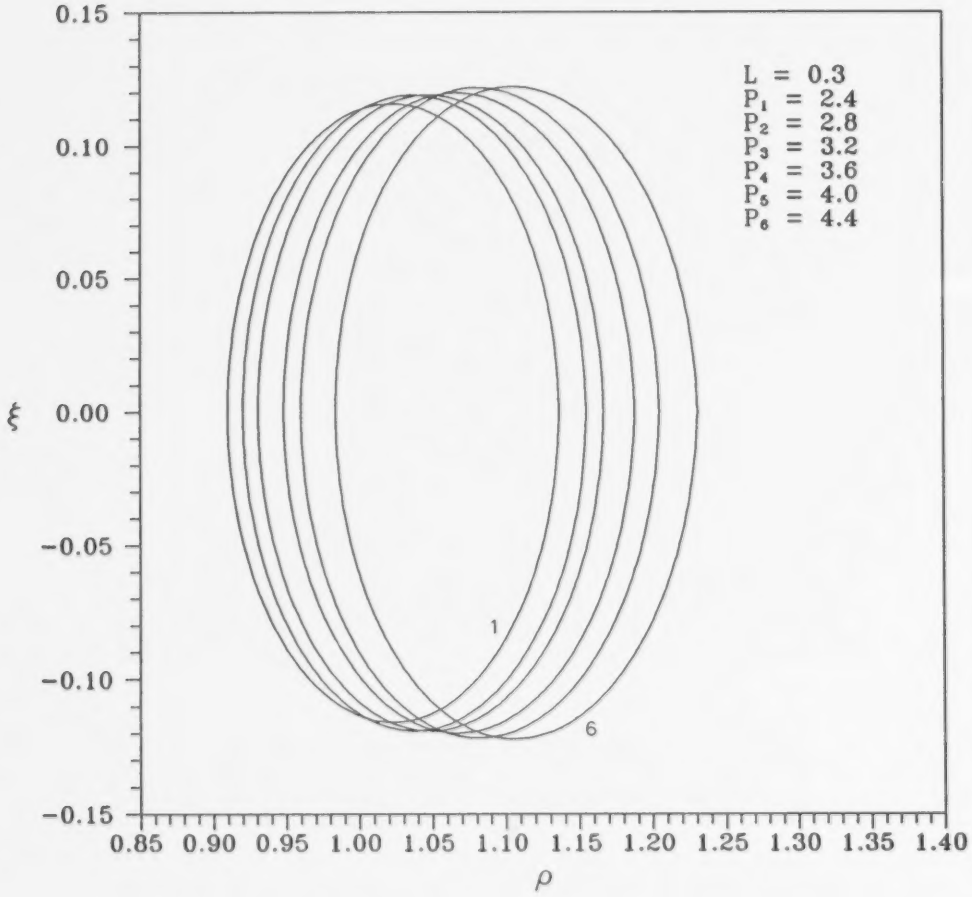

Fig. 3.9.9 Deformed axial cross-sections

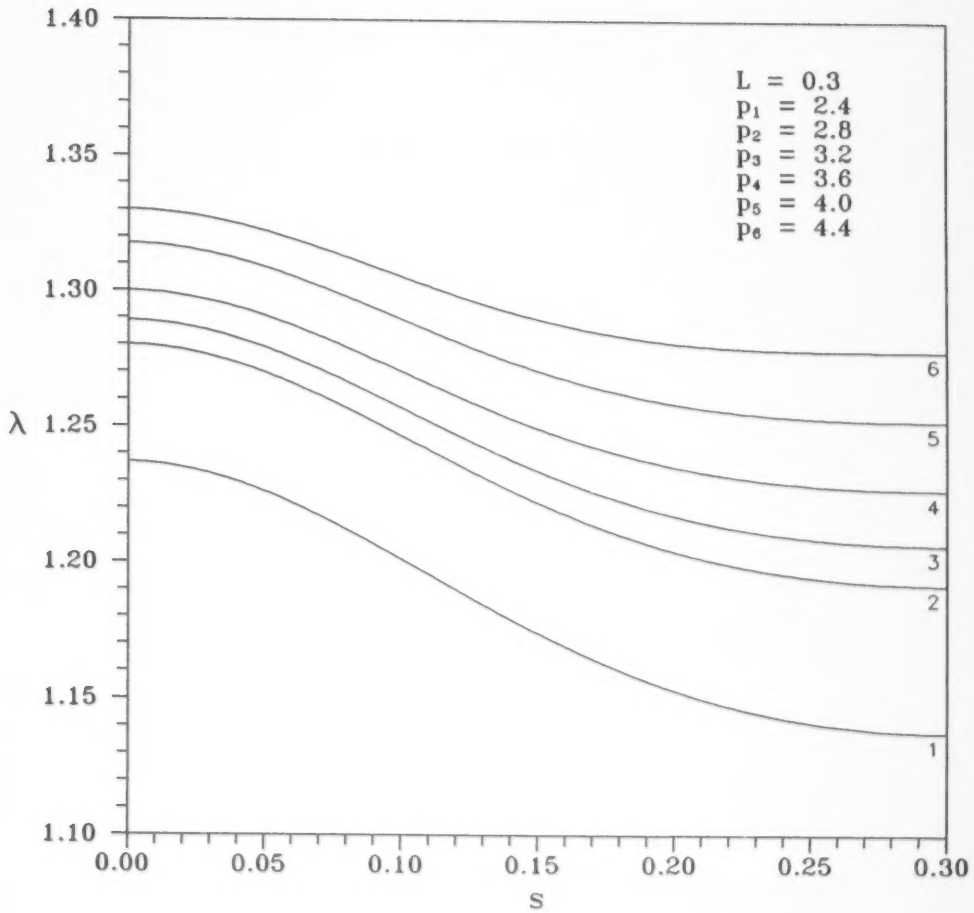

Fig. 3.9.10 Distribution of λ

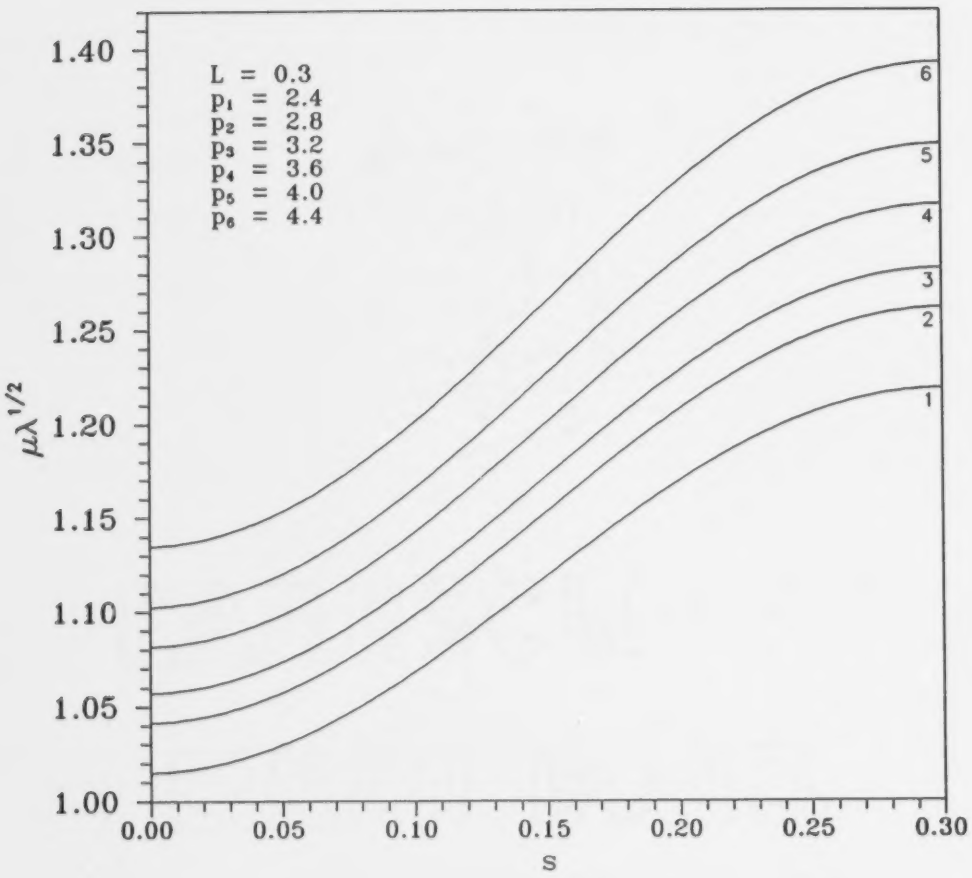

Fig.3.9.11 Distribution of $\mu\lambda^{1/2}$

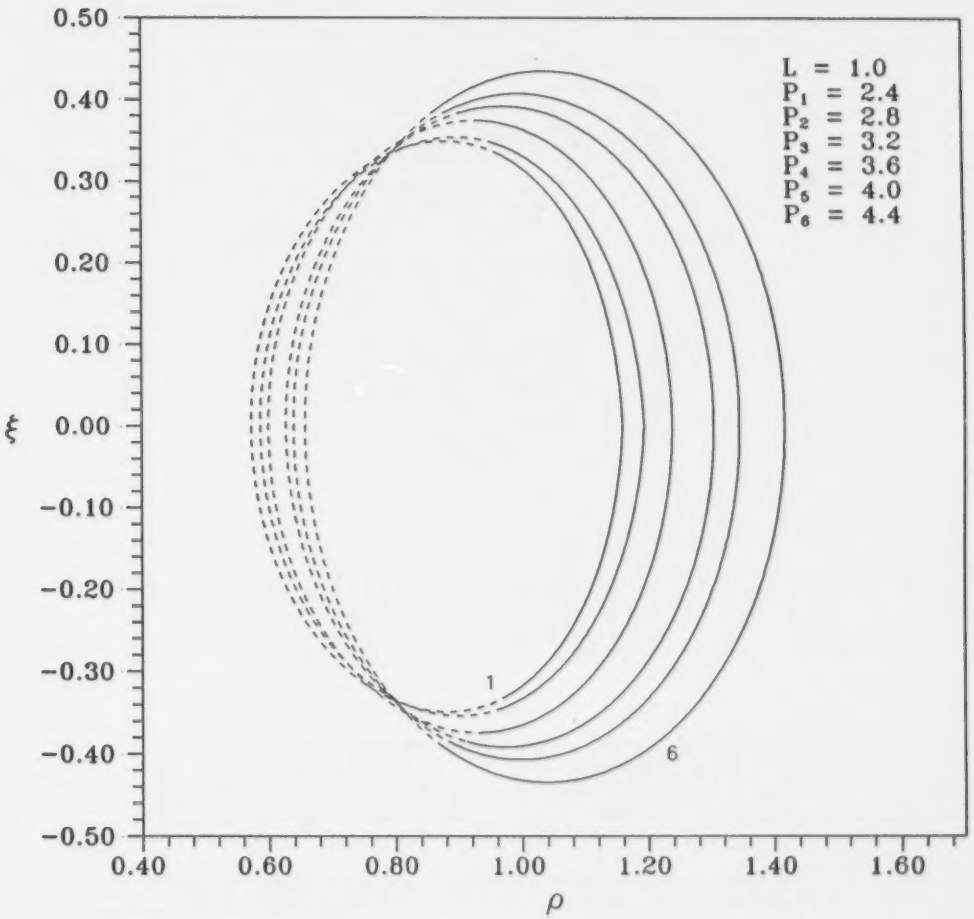

Fig.3.9.12 Deformed axial cross-sections

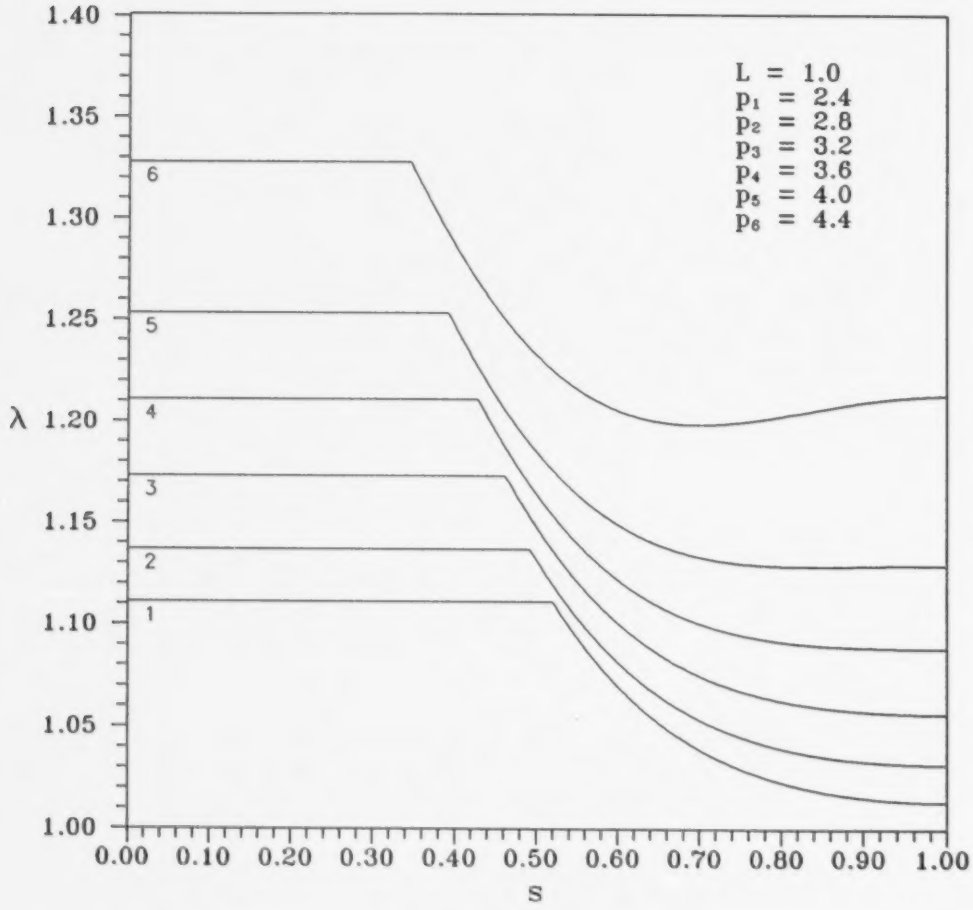

Fig.3.9.13 Distribution of λ

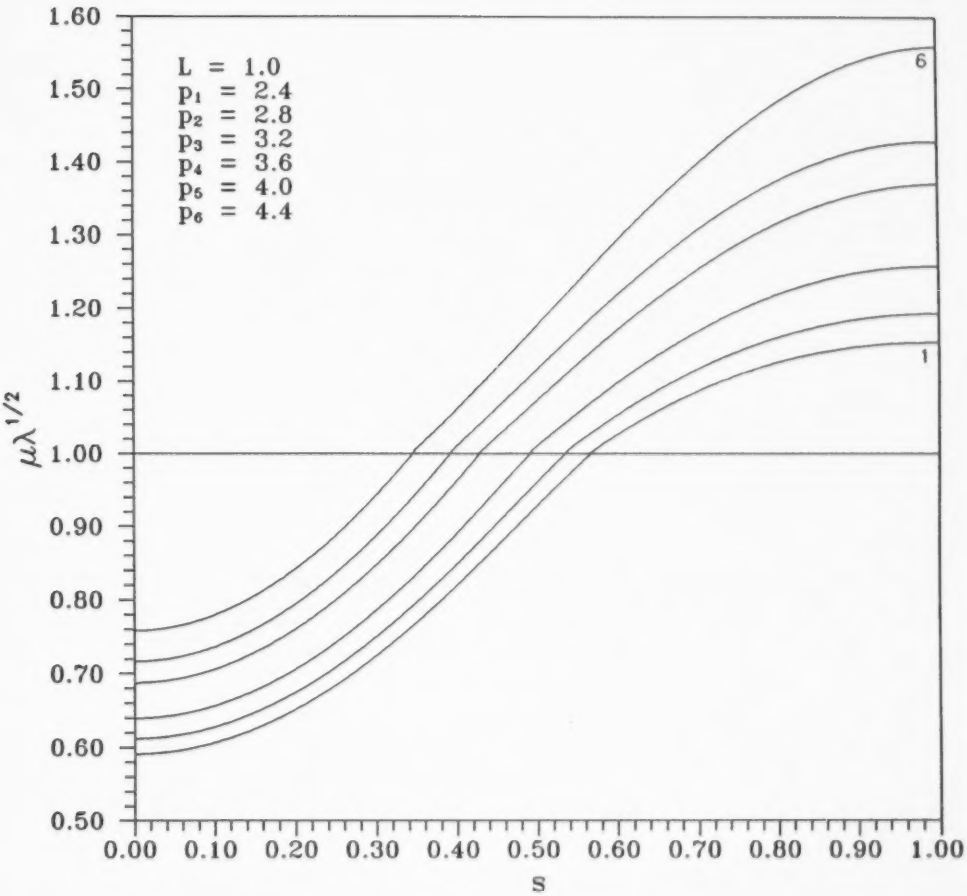

Fig.3.9.14 Distribution of $\mu\lambda^{1/2}$

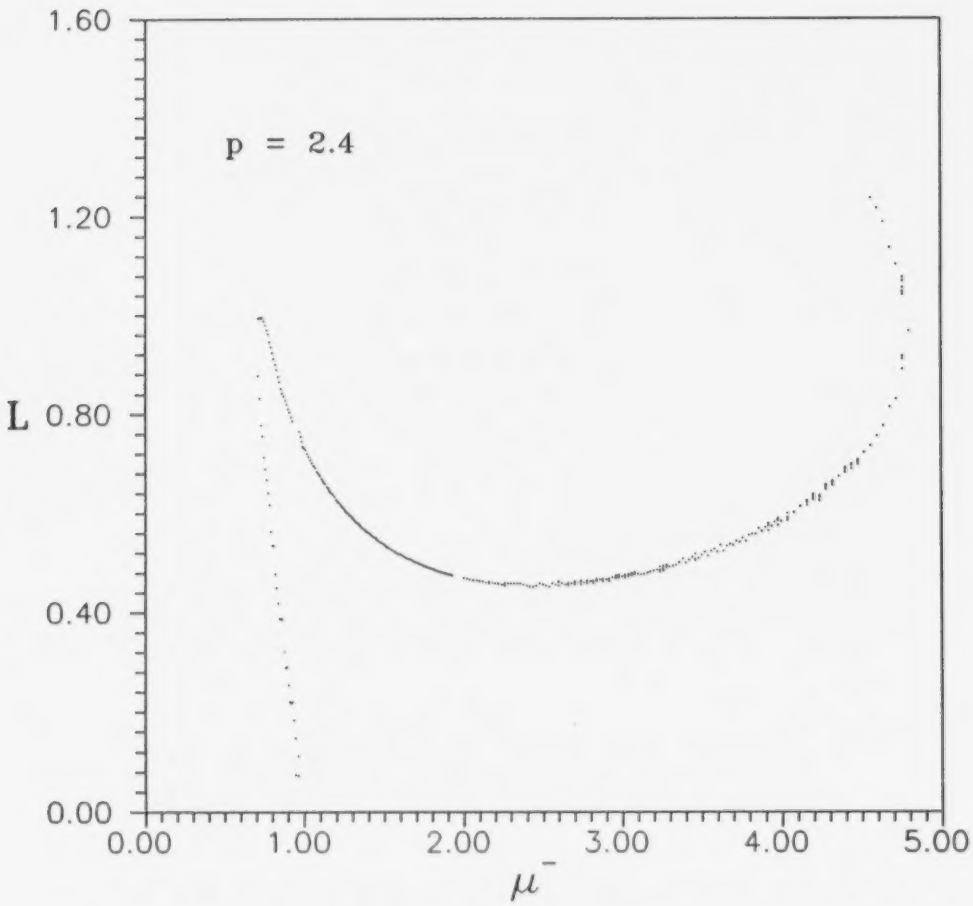

Fig. 3.9.15 Plot of the reference height L against the initial guess μ^-.

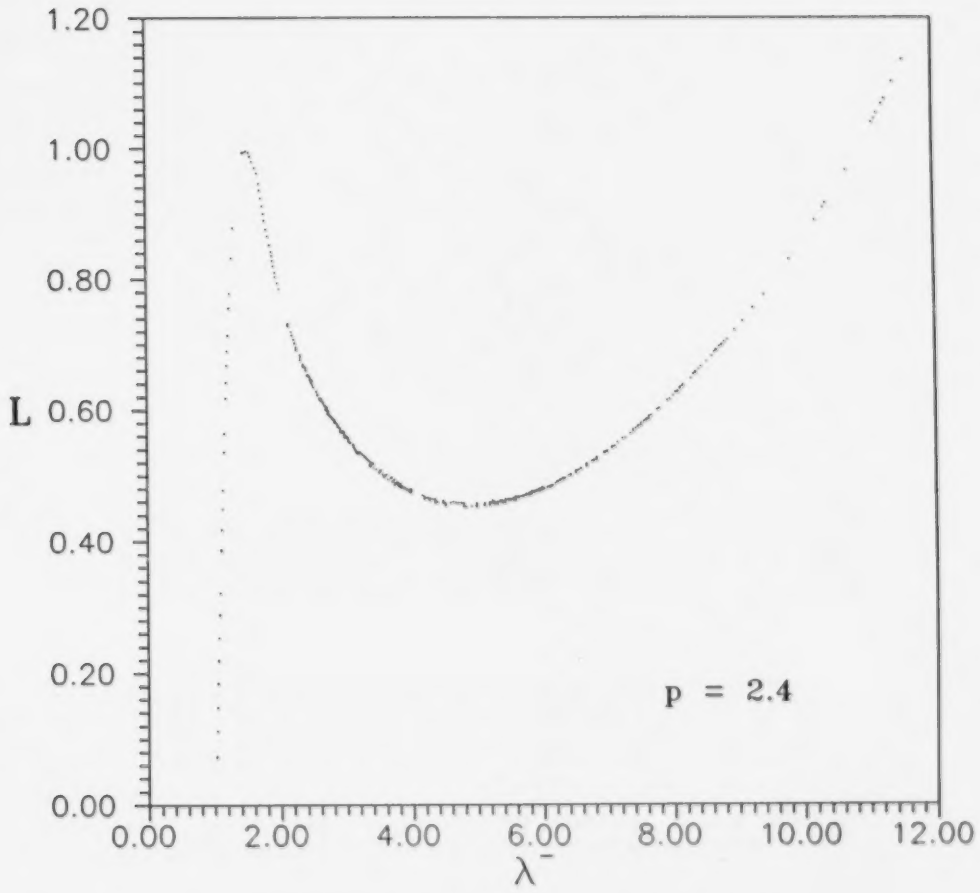

Fig.3.9.16 Plot of the reference height L against the initial guess λ^-.

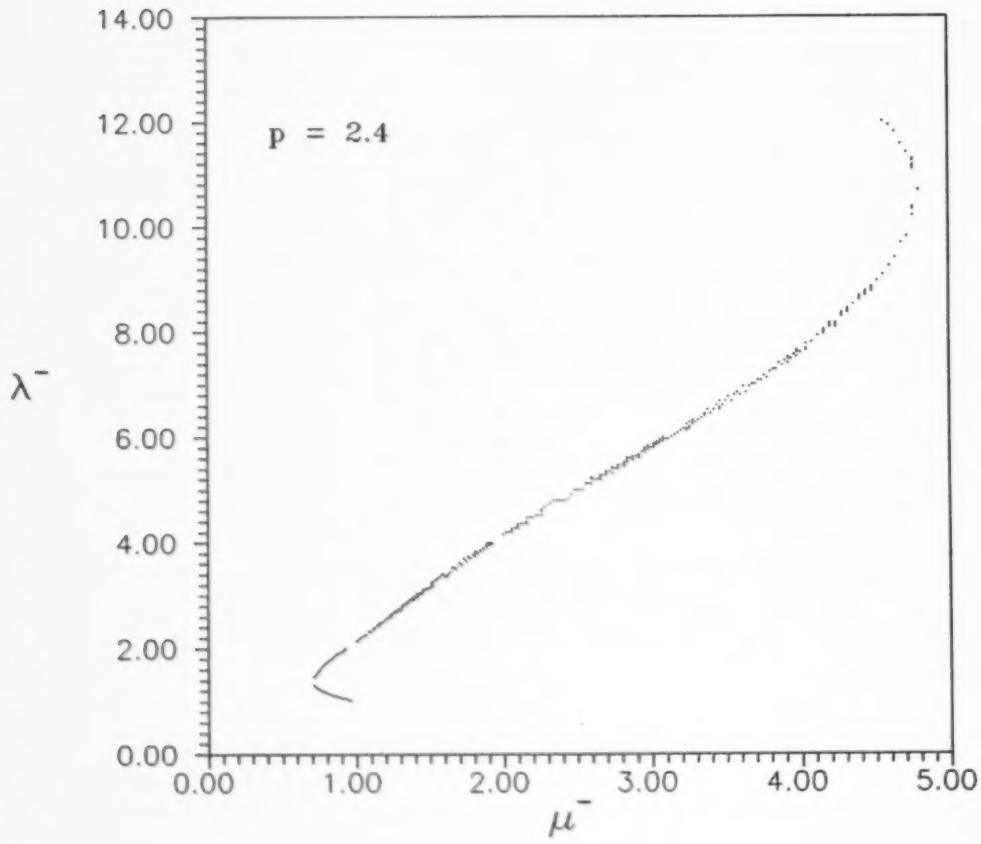

Fig.3.9.17 Solutions of λ^- and μ^-.

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Appendix A

As discussed in section 2.4, to facilitate the calculations, we use λ instead of r as independent variable in the formulations of the problem. In this appendix, we demonstrate the procedure by using equations (2.4.2, 3) to convert $\phi(r)$ and $\delta(r)$ to functions of the stretch λ. In the following process, for the sake of notational convenience, we use the same symbols to denote functions of λ and r. Different functions are distinguished by exhibiting the arguments explicitly.

By substituting $\sec\gamma$ obtained from $(2.4.2)_1$ into the integrand in $(2.4.9)_1$ we have the integral

$$\phi(\lambda) = \frac{1}{a}\int_{\lambda_i}^{\lambda} x\, f(x) r(x) r'(x)\, dx \ , \tag{A.1}$$

where $r(\lambda)$ is the positive root of $(2.4.3)_1$ and $\lambda_i = \lambda(r_i)$ is the stretch to be specified at the inner boundary r_i. If r is regarded as the function of λ, we obtain $dr/d\lambda$ from $(2.4.3)_1$. It is

$$rr'(\lambda) = -(a^2/f^3)f'(\lambda) - b^2/\lambda^3 \ . \tag{A.2}$$

Upon substituting (A.2) into (A.1) and integrating by parts, $\phi(\lambda)$ is

$$\phi(\lambda) = a\left[\frac{x}{f(x)}\Big|_{\lambda_i}^{\lambda} - \int_{\lambda_i}^{\lambda}\frac{dx}{f(x)}\right] - \frac{b^2}{a}\int_{\lambda_i}^{\lambda}\frac{f(x)}{x^2}dx \ . \tag{A.3}$$

The remaining integral can be integrated analytically for the $f(\lambda)$ given by $(2.2.3)_1$. Substituting $(2.2.3)_1$ into the above integral, we have derived an explicit expression for

$\phi(\lambda)$ in terms of λ. It is

$$\phi(\lambda) = \left(\frac{a}{2G}\right)\left[\frac{x^2}{(x^2-1)} - \frac{1}{2}\ln(x^2-1)\right]\Bigg|_{\lambda_1}^{\lambda} - \left(\frac{2Gb^2}{a}\right)\left[\ln x + \frac{1}{2x^2}\right]\Bigg|_{\lambda_1}^{\lambda}. \quad (A.4)$$

This completes the conversion of $\phi(r)$ into the function of λ.

To convert the function $\delta(r)$, we first use $(2.4.3)_2$ to replace $\tan\gamma$ in $(2.4.10)_2$ to write

$$\delta(r) = \frac{b}{a}\int_{r_1}^{r}\frac{f(\lambda)}{x\,\lambda}dx\,. \quad (A.5)$$

Integrating by parts results in an expression like

$$\delta(r) = \frac{b}{a}\left\{\left[\frac{f(\lambda)}{\lambda}\ln x\right]\Bigg|_{r_1}^{r} - \int_{r_1}^{r}\ln x\, d\left(\frac{f}{\lambda}\right)\right\} \quad (A.6)$$

Again we use $(2.4.3)_1$ to eliminate r. To this end, we derive the relation

$$\ln r = \ln(a/f) + \tfrac{1}{2}\ln(1 + \lambda^{-2}f^2b^2/a^2). \quad (A.7)$$

Use of (A.7) in the integral in (A.6) leads to

$$\int \ln\left(\frac{a}{f}\right)d\left(\frac{fb}{a\lambda}\right) + \frac{1}{2}\int \ln\left(1 + \frac{f^2b^2}{a^2\lambda^2}\right)d\left(\frac{fb}{a\lambda}\right), \quad (A.8)$$

The first integral can be integrated again and it gives

$$\frac{b}{a}\left\{\left[\frac{f(x)}{x}\ln\left(\frac{a}{f(x)}\right)\right]\Bigg|_{\lambda_1}^{\lambda} + \int_{\lambda_1}^{\lambda}\frac{f'(x)}{x}dx\right\}. \quad (A.9)$$

By variable substitution, evaluation of the second integral of (A.8) is just a standard exercise.

Adding this result together with (A.7) and (A.9), we reduce (A.6) to the form

$$\delta(\lambda) = \frac{b}{a}\left[\frac{f(x)}{x}\Big|_{\lambda_1}^{\lambda} - \int_{\lambda_1}^{\lambda} \frac{f'(x)}{x}\,dx\right] - \tan^{-1}\left[\frac{f(x)b}{x\,a}\Big|_{\lambda_1}^{\lambda}\right]. \tag{A.10}$$

Evaluating the second integral of (A.10) by parts, (A.10) becomes

$$\delta(\lambda) = \frac{b}{a}\int_{\lambda_1}^{\lambda} \frac{f(x)}{x^2}\,dx + \tan^{-1}\left[\frac{f(x)b}{a\,x}\Big|_{\lambda_1}^{\lambda}\right]. \tag{A.11}$$

It may be seen that the first of (A.11) is the same as the second of (A.3). Consequently, for the $f(\lambda)$ defined in (2.2.3), it can be integrated explicitly. We finally obtain an explicit expression for $\delta(\lambda)$ in the form

$$\delta(\lambda) = \left(\frac{2Gb}{a}\right)\left(\ln x + \frac{1}{2x^2}\right)\Big|_{\lambda_1}^{\lambda} + \tan^{-1}\left[\left(\frac{2Gb}{a}\right)\left(1 - \frac{1}{x^2}\right)\Big|_{\lambda_1}^{\lambda}\right]. \tag{A.12}$$

Appendix B

As we have remarked in section 3.5 that in treating the point-load problems which inherently possess singularity at the pole, i.e. stress is singular at the pole, the solution of the resulting integral equation (3.4.17) exists only if the relaxed strain energy $w^R(\lambda)$ satisfies a certain growth condition as the singular point is approached, i.e. existence of solutions depends on the functional form of the relaxed strain energy function associated with wrinkled membranes. For convenience, we recite the first integral in (3.4.17) by

$$\int_{\lambda_1}^{\infty} \frac{\lambda w^R{}_{\lambda\lambda}}{w^R_\lambda \sqrt{(w^R_\lambda)^2 - c^2}}\,d\lambda \tag{B.1}$$

As suggested by (B.1), stress resultant $w^R{}_\lambda(\lambda)$ must increase more rapidly than linearly

in λ to ensure the existence of solution. In this appendix, we demonstrate that relaxed strain energy function (3.1.9) associated with Ogden's material fulfils the required growth condition, but fails for other material such as Neo-Hookean, Mooney-Rivlin, etc. In particular, for illustrative examples, we first discuss Ogden's material which we have used in section 3.4 to obtain solutions for the point-load problems. Secondly, we briefly demonstrate that material of Neo-Hookean type delivers no solution in the case of point-load. Qualitatively, the conclusions that we have arrived at agree with that of Fulton's [3.14].

A. *Ogden's material*

For Ogden's material, the adapted membrane strain energy function is given in (3.1.3). When we choose the natural width (3.1.8), the relaxed strain energy $w^R(\lambda)$ is derived and expressed by (3.1.9). Definition of the relaxed strain energy dictates that $w^R(\lambda)$ is function of λ alone. Their derivatives are

$$w_\lambda^R(\lambda) = \sum_{p=1}^{3} g_p\left[\lambda^{(\alpha_p-1)} - \lambda^{(\frac{\alpha_p}{2}+1)}\right]; \tag{B.2}$$

$$w_{\lambda\lambda}^R(\lambda) = \sum_{p=1}^{3} g_p\left[(\alpha_p-1)\lambda^{(\alpha_p-2)} + (\frac{\alpha_p}{2}+1)\lambda^{(\frac{\alpha_p}{2}+2)}\right] \tag{B.3}$$

The standard treatment of integrations whose integrants consisting of the isolated singular points is to make a change of independent variable [3.16]. More than often, such a change of independent variable will also give results in a more satisfactory form to conduct the quantitative behaviour analysis of the integrants. let $\lambda = u^{-1}$; (B.1)

becomes

$$w_\lambda^R(u) = \sum_{p=1}^{3} g_p \left[u^{(1-\alpha_p)} - u^{\left(\frac{\alpha_p}{\gamma}+1\right)} \right] \tag{B.4}$$

note that the singularity in the form of a pole of order $|1 - \alpha_p|$ if the exponent $(1 - \alpha_p) < 0$, e.g. the first term in the bracket of (B.3) becomes infinity as $u \to 0$. This is true by considering the set of elastic constants specified in (3.1.2). The second term in the square bracket remains regular. Using the technique of dividing out the singularity, we write $w^R(1/u) = f(1/u)g(1/u)$, where $f(1/u)$ is a function that has a singularity of the same kind as that of $w^R(1/u)$, whereas $g(1/u)$ is a regular function in the domain concerned. By factoring out the leading term $1/u^{(\alpha_{max}-1)}$ (as $u \to 0$), we have

$$\begin{aligned} w_\lambda^R(u) &= \sum_{p=1}^{3} g_p \frac{u^{(1-\alpha_p)} - u^{\left(\frac{\alpha_p}{\gamma}+1\right)}}{u^{(\alpha_p^{max}-1)}} \cdot u^{(\alpha_p^{max}-1)} \\ &= \left[\frac{1}{u^{(\alpha_p^{max}-1)}} \right] \sum_{p=1}^{3} g_p \left\{ u^{(-\alpha_p+\alpha_p^{max})} - u^{\left(\frac{\alpha_p}{\gamma}+\alpha_p^{max}\right)} \right\} \end{aligned} \tag{B.5}$$

thus, functions $f(\cdot)$ and $g(\cdot)$ are, respectively,

$$\begin{aligned} f(u) &= \frac{1}{u^{(\alpha_p^{max}-1)}} \\ g(u) &= \sum_{p=1}^{3} g_p \left\{ u^{(\alpha_p^{max}-\alpha_p)} - u^{\left(\alpha_p^{max}+\frac{\alpha_p}{\gamma}\right)} \right\} \end{aligned} \tag{B.6}$$

Note that $g(\cdot)$ is regular and becomes zero as $u \to 0$.

Change of independent variable from λ to u transforms (B.1) into the following form. Note the fact that $u \to 0$ as $\lambda \to \infty$, we have

$$\int_0^{\frac{1}{\lambda_b}} \frac{w^R{}_{\lambda\lambda}(u)}{u^3\, w_\lambda^R(u)\sqrt{(w_\lambda^R(u))^2 - c^2}}\, du \tag{B.7}$$

Substitute (B.5, 6) into the denominator of the integrant in (B.7), the corresponding denominator becomes

$$\begin{aligned} & u^3 \cdot f(u) \cdot g(u) \cdot \sqrt{f(u)^2 \cdot g(u)^2 - c^2 \cdot f(u)^2 \cdot f(u)^{-2}} \\ &= u^3 \cdot f(u)^2 \cdot g(u) \cdot \sqrt{g(u)^2 - c^2 u^{2(\alpha_p^{max}-1)}} \\ &= u^{(5-2\alpha_p^{max})} \cdot g(u) \cdot \sqrt{g(u)^2 - c^2 u^{2(\alpha_p^{max}-1)}}\ ; \end{aligned} \tag{B.8}$$

Replacing λ by u^{-1} in (B.3) and diving the term $u^{(5-2\alpha pmax)}$, the nominator of (B.7) is

$$\begin{aligned} w_{\lambda\lambda}^R(u) \cdot u^{(2\alpha_p^{max}-5)} &= \sum_{p=1}^{3} g_p \left\{ (\alpha_p - 1) u^{(2-\alpha_p)} + (\frac{\alpha_p}{2} + 1) u^{(\frac{\alpha_p}{2}+2)} \right\} \cdot u^{(2\alpha_p^{max}-5)} \\ &= \sum_{p=1}^{3} g_p \left\{ (\alpha_p - 1) u^{(2\alpha_p^{max}-\alpha_p-3)} + (\frac{\alpha_p}{2} + 1) u^{(2\alpha_p^{max}+\frac{\alpha_p}{2}-3)} \right\} \end{aligned} \tag{B.9}$$

For $\alpha_p{}^{max} = 5$ (see 3.1.2), (B.9) decays to zero as $u \to 0$.

Finally, using h(u) to denote the left-hand side function in (B.9), and substituting (B.8, 9) into (B.7), we have an integral

$$\int_0^{\frac{1}{\lambda_b}} \frac{h(u)}{g(u) \cdot \sqrt{g(u)^2 - c^2 u^{2(\alpha_p^{max}-1)}}}\, du \tag{B.10}$$

It can be seen that the integrant is continuous in the interval $(0, \lambda_b)$ and zero at the end-points $u = 0,\ 1/\lambda_b$. Thus the integration of (b.10) is finite and involves no singularity.

B. *Neo-Hookean material*

We now consider the so-called Neo-Hookean material. Strain energy function adapted

for use in the membrane theory is given by

$$w/2G = \lambda^2 + \mu^2 + \lambda^{-2}\mu^{-2} - 3 \qquad (B.11)$$

where G is the material constant associated with the infinitesimal deformations. For the axisymmetric deformations of surfaces of revolution, λ may denote the meridional stretch ,while μ denotes the circumferential stretch. To determine the deformation of a surface that is wrinkled, relaxed strain energy function must be used in the governing equations. For the problems considered in previous sections, the deformed membrane is wrinkled circumferentially, thus the associated relaxed strain energy function can be derived from (B.11) by setting $\mu = \bar{v}(\lambda)$, where $\bar{v}$ is so-called natural width. Using $\bar{v}(\lambda) = \lambda^{-1/2}$ (see 3.1.8), the relaxed strain energy function is

$$w^R(\lambda)/2G = \lambda^2 + 2\lambda^{-1} - 3 \qquad (B.12)$$

To proceed, we need the first and second derivatives of $w^R(\cdot)$. They are given by

$$w^R{}_\lambda(\lambda)/2G = 2(\lambda - \lambda^{-2}) \qquad (B.13)$$

$$w^R{}_{\lambda\lambda}(\lambda)/2G = 2(1 + 2\lambda^{-3}) \qquad (B.14)$$

Introduce the change of variable from λ to u by $\lambda = u^{-1}$, (B.13, 14) become

$$w^R{}_\lambda(u)/2G = 2(u^{-1} - u^2) \qquad (B.14)$$

$$w^R{}_{\lambda\lambda}(u)/2G = 2(1 + 2u^3) \qquad (B.15)$$

Evidently (B.14) is singular at $u = 0$ in the form of $1/u$. We write (B.14) in the following form

$$w^R{}_\lambda(u)/2G = f(u)g(u) \qquad (B.16)$$

where $f(u) = u^{-1}$, $g(u) = 2(1 - u^3)$. At the lower limit, $g(0) = 2$ and is finite.

Substituting (B.16) into the denominator of the integrant of (B.7), it becomes

$$
\begin{aligned}
& u^3 \cdot f(u) \cdot g(u) \cdot \sqrt{f(u)^2 \cdot g(u)^2 - c^2 \cdot f(u)^2 \cdot f(u)^{-2}} \\
&= u^3 \cdot u^{-2} \cdot g(u) \cdot \sqrt{g(u)^2 - c^2 u^2} \\
&= u \cdot g(u) \cdot \sqrt{g(u)^2 - c^2 u^2} \ ;
\end{aligned}
\tag{B.17}
$$

Finally, substitute (B.15) and (B.17) into (B.7), the integral becomes

$$
\int_0^{\frac{1}{\kappa}} \left[\frac{1}{u}\right] \left[\frac{2(1+2u^3)}{g(u) \cdot \sqrt{g(u)^2 - c^2 u^2}}\right] du \tag{B.18}
$$

At the lower limit $u = 0$, the term in the second square bracket approaches its limit value 1/2, whereas the first square bracket becomes infinity. This is an unremovable singular point in the form of a pole of order 1. Thus the relaxed strain energy function expressed in terms of principal stretch of the quadratic order delivers no solution to the integral (B.7).